Science Olympiad
Machines C
BEARSO Invitational

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Section B Solutions

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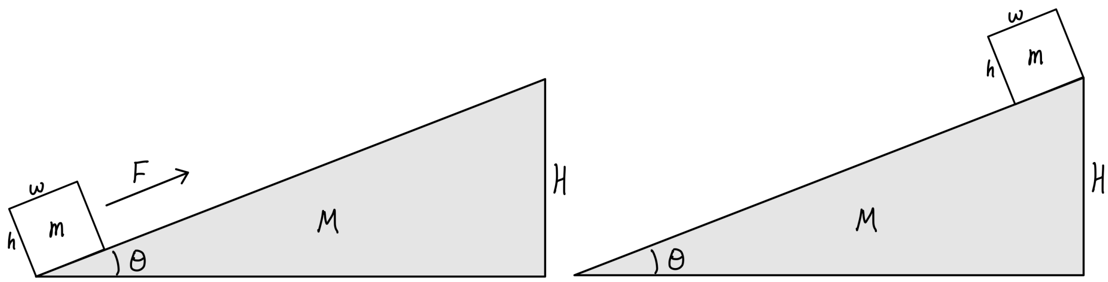
Feedback? **Test Code**: *2021BEARSO-MachinesC-Shell*

Section B: Free Response

Points are shown for each question or sub-question, for a total of 120 points.

1. (24 points) An inclined plane of angle $\theta = 25°$, height $H = 4\,\text{m}$, and mass $M = 20\,\text{kg}$ is locked to the floor. A block of mass $m = 3\,\text{kg}$, width $w = 50\,\text{cm}$, and height $h = 30\,\text{cm}$ lies on the end of the plane and is pushed up the plane. Both objects are of uniform density. Below is a diagram of the system initially (on the left) and the system after the block has been raised (on the right).



(a) Assuming the inclined plane and block are frictionless, find:
   i. (2 points) The force required to keep the block at rest, in N.
   ii. (2 points) The energy required to lift the block to the top of the plane (right diagram), in J.

(b) In reality, the system is not ideal. It takes 150 J of work to lift the block from the bottom of the plane (left diagram) to the top of the plane (right diagram).
   i. (3 points) Calculate the AMA of the ramp.
   ii. (2 points) One possible reason for this energy loss is air resistance. Name another likely reason for this energy loss.
   iii. (3 points) If the primary energy loss occurred through air resistance, describe how it can be minimized.

(c) Now let's return to the original assumption of ideal conditions: the block, plane, and floor are all frictionless. Once the block is lifted to the top of the plane (right diagram), both the block and inclined plane are simultaneously released, so that both can slide freely on each other and on the floor.
   i. (3 points) Qualitatively describe what happens to the system.
   ii. (4 points) Calculate the acceleration of the inclined plane right after the release, in $\text{m}\,\text{s}^{-2}$.
   iii. (5 points) Find the velocity of the block a long time after the release, in $\text{m}\,\text{s}^{-1}$.

Solution:

(a) i. Use $\Sigma F = ma$ parallel to the inclined plane surface.

$$\Sigma F_{\|} = 0 \implies F - mg\sin\theta = 0 \implies F = 3\,\text{kg} \times 9.81\,\text{m}\,\text{s}^{-2} \times \sin 55° = 12.4\,\text{N}$$

ii. Find the distance the block travels and multiply by the force found in (a) to get work.

$$W = Fd = F \times \left(\frac{H}{\sin\theta} - w\right) = 112\,\text{J}$$

Partial credit was given to teams who omitted the width of the block.

(b) i.

$$\text{AMA} = \eta \times \text{IMA} = \frac{W_o}{W_i} \times \text{IMA} = \frac{112\,\text{J}}{150\,\text{J}} \times \frac{1}{\sin 25°} = 1.77$$

ii. Friction, sound, etc.

iii. The block can be raised slower, since air resistance is proportional to velocity. Also, the block could be modified to be more aerodynamic by decreasing its surface area.

(c) i. The block begins to slide down the plane to the left due to gravity and the plane begins to slide to the right due to the normal force from the block. Once the block reaches the floor, they move away from each other.

ii. Let the normal force between the block and the plane be N and the accelerations of the blocks (in their respective axes) be a_{mx}, a_{my}, and a_{Mx}. We relate these variables with 4 equations and solve for a_{Mx}. Up and right are positive.

$$\begin{cases} -N\sin\theta = ma_{mx} \\ N\sin\theta = Ma_{Mx} \\ N\cos\theta - mg = ma_{my} \\ \tan\theta = -a_{my}/a_{Mx} \end{cases} \implies a_{Mx} = \frac{mg}{M\cot\theta + m\tan\theta} = 0.664\,\text{m}\,\text{s}^{-2}$$

iii. Once the block is released, the gravitational potential energy is converted into kinetic energy of the two objects. However, momentum is conserved. To calculate the potential energy of the block, find the distance its center of mass falls. There is some tricky geometry that has been omitted from this solution. It ultimately contributes a negligible ϵ.

$$U = mg(\Delta y) = mg \times (H + \epsilon - \frac{h}{2})$$

The two conservation of energy and momentum equations constrain the solution.

$$\begin{cases} mg \times (H + \epsilon - \frac{h}{2}) = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2 \\ mv_m = Mv_M \end{cases}$$

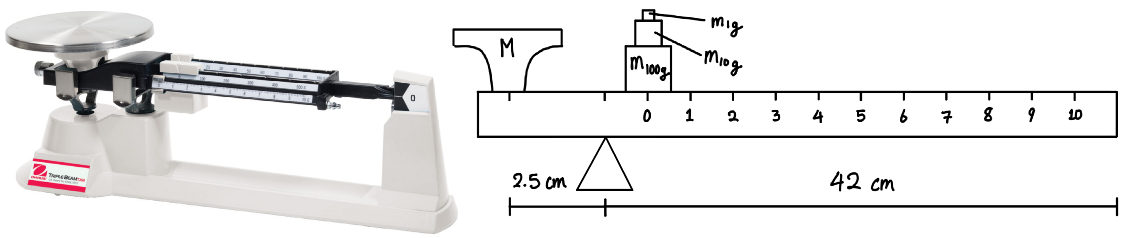
Plugging into the equations, we get

$$\begin{cases} 3\,\text{kg} \times 9.81\,\text{m}\,\text{s}^{-2} \times (4 + 0.0303 - 0.15)\ \text{m} = 1.5\,\text{kg} \times v_m^2 + 10\,\text{kg} \times v_M^2 \\ 3\,\text{kg} \times v_m = 20\,\text{kg} \times v_M \end{cases}$$

$$v_m = 8.14\,\text{m}\,\text{s}^{-1} \text{ and } v_M = 1.22\,\text{m}\,\text{s}^{-1}$$

2. (12 points) A triple beam balance is an instrument used to measure mass very precisely. Shown below is a picture of it (left) and a lever representation of it (right). There are 3 riders ($m_{100\,\mathrm{g}}$, $m_{10\,\mathrm{g}}$, and $m_{1\,\mathrm{g}}$), a weighing pan of mass M, and a lever with linear density $\lambda = 1\,\mathrm{g\,cm^{-1}}$. All gaps to the right of the fulcrum are of equal length.

To weigh an object, it is placed on the weighing pan. Then, the riders are adjusted such that the balance is in equilibrium. The final locations of the riders depends on the mass of the object. For example, if a 461 g object is weighed, the $m_{100\,\mathrm{g}}$, $m_{10\,\mathrm{g}}$, and $m_{1\,\mathrm{g}}$ riders will be placed over the ticks marked 4, 6, and 1, respectively, so that the balance is in equilibrium. Assume the center of mass of the weighing pan and the weighed objects are above the left tick mark.



(a) Find, in grams:

i. (3 points) $m_{100\,\mathrm{g}} + m_{10\,\mathrm{g}} + m_{1\,\mathrm{g}}$. *(Hint: The mass of m_{100g}, m_{10g}, and m_{1g} are not 100 g, 10 g, and 1 g. What does it means to move the masses by one tick mark?)*

ii. (4 points) M.

(b) There is a frictional moment of $2.5 \times 10^{-6}\,\mathrm{N\,m}$ at the fulcrum.

i. (3 points) What is the reading error of the balance, in grams?

ii. (2 points) How could the balance be modified to lower its error?

Solution:

(a) i. The current setup of the balance is in equilibrium, only when mass is added to the weighing pan, must the masses move to reach an equilibrium point. So, if 100 g, 10 g, or 1 g are added, the respective riders must move by one tick (of length $42/12 = 3.5\,\text{cm}$) and provide the equal and opposite amount of torque.

$$\begin{cases} \Delta T_{100\text{g}} = m_{100\text{g}}\, g \times 3.5\,\text{cm} \implies m_{100\text{g}} = \frac{2.5}{3.5} \times 100\,\text{g} = 71.4\,\text{g} \\ \Delta T_{10\text{g}} = m_{10\text{g}}\, g \times 3.5\,\text{cm} \implies m_{10\text{g}} = \frac{2.5}{3.5} \times 10\,\text{g} = 7.14\,\text{g} \\ \Delta T_{1\text{g}} = m_{1\text{g}}\, g \times 3.5\,\text{cm} \implies m_{1\text{g}} = \frac{2.5}{3.5} \times 1\,\text{g} = 0.714\,\text{g} \end{cases}$$

$$m_{100\text{g}} + m_{10\text{g}} + m_{1\text{g}} = 79.3\,\text{g}$$

ii. We can balance the torques on the lever, using the rider masses solved in (a.i), to solve for M. Notice that there is a torque from the weight of the lever itself, as the center of mass of the lever is not at the fulcrum.

$$Mg \times 2.5\,\text{cm} = (m_{1\text{g}} + m_{10\text{g}} + m_{100\text{g}})g \times 3.5\,\text{cm} + \tau_{\text{lever}}$$

$$\tau_{\text{lever}} = \lambda l \times x_{\text{com}} = 44.5\,\text{g} \times 19.75\,\text{cm}$$

$$\therefore M = 463\,\text{g}$$

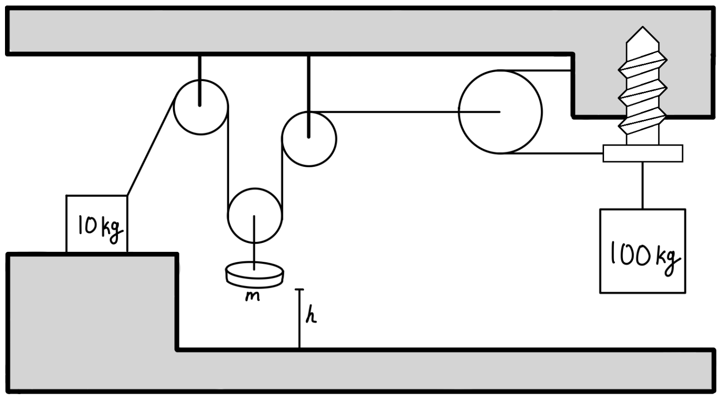
(b) i. Again, balance the torques for some Δm, which acts as the error in the balance.

$$2.5 \times 10^{-6}\,\text{N m} = \Delta m g \times 0.025\,\text{m} \implies \Delta m = 1.02 \times 10^{-5}\,\text{kg}$$

So the reading error is $\pm 0.01\,\text{g}$.

ii. Various answers accepted: decrease the total weight on the lever (by lengthening the lever, moving the zero point towards the fulcrum, etc.), decrease the contact area at the fulcrum, change the fulcrum and lever material with lower μ, etc.

3. (12 points) A compound machine is shown below. It consists of a frictionless and massless single-threaded screw, an ideal pulley system with two fixed and two movable pulleys connected with lightweight, flexible cords, a 10 kg block on a rough surface, a 100 kg block attached to the bottom of the screw, and a disk of mass m lifted a distance of $h = 2$ m off the ground. The screw has a 7 mm pitch, a 3 cm shaft radius, and a 5 cm cap radius where a cord is attached to. The aforementioned cord is attached to the edge of the cap radius such that pulling on it tightens the screw. The machine is currently in equilibrium.



(a) (2 points) What is the mass of the disk, in kg? *(Hint: Treat the block as an immovable surface.)*

(b) (3 points) If the cord is connected to the 10 kg block at a 55° angle with respect to the ground, what are the possible values for the coefficient of static friction between the block and the floor?

(c) (4 points) The cord connected to the 100 kg block is cut. What is the upwards velocity of the screw right when the disk has fallen half the distance to the ground, in $\text{cm}\,\text{s}^{-1}$?

(d) (3 points) How far upwards would the screw have moved from its initial position once the disk hits the ground, in cm?

Solution:

(a) Assume the 10 kg block is an immovable surface and that the tension in the cords are just enough to hold the 100 kg block and disk up. Let us set T_1 to be the tension in the cord attached to the 10 kg block and T_2 to be the tension in the cord attached to the screw. Using $\Sigma F = ma$ on the two movable pulleys we find

$$2T_1 = mg \text{ and } T_1 = 2T_2$$

Also, T_2 can be defined as

$$T_2 \times \text{IMA} = 100\,\text{kg} \times g$$

where IMA is

$$\text{IMA} = \frac{2\pi \times 50\,\text{mm}}{7\,\text{mm}} = 44.9$$

Now we can solve for m,

$$m = \frac{4 \times 100\,\text{kg}}{44.9} = 8.91\,\text{kg}$$

(b) For the 10 kg block to stay at rest, static friction must be greater than the horizontal component of the tension. We can solve for this using $\Sigma F = ma$.

$$\Sigma F_y = 0 \implies N + T_1 \sin\theta - 10\,\text{kg} \times g = 0 \implies N = 10\,\text{kg} \times g - T_1 \sin\theta$$

$$\Sigma F_x = 0 \implies N\mu_s \geq T_1 \cos\theta \implies \mu_s \geq \frac{T_1 \cos\theta}{10\,\text{kg} \times g - T_1 \sin\theta}$$

We know from (a) that $2T_1 = 8.91\,\text{kg} \times g$, so plugging in values and solving for μ_s results in

$$\mu_s \geq 0.403$$

(c) Since the screw is massless and the 10 kg block does not move, we can treat this problem as a simple energy conservation setup.

$$mgh = \frac{1}{2}mv_d^2 \implies v_d = \sqrt{2gh} = 443\,\text{cm}\,\text{s}^{-1}$$

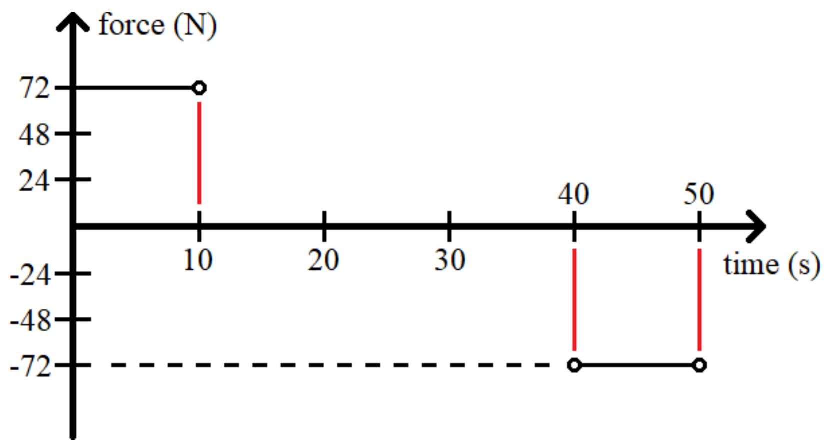
$$v_s = \frac{4}{\text{IMA}} \times v_d = 39.5\,\text{cm}\,\text{s}^{-1}$$

(d) The same relationship between the disk and the screw can be applied again to the displacement of the screw.

$$\Delta x_s = \frac{4}{\text{IMA}} \times 200\,\text{cm} = 17.8\,\text{cm}$$

4. (21 points) An elevator is designed with a counterweight system, connected together with a lightweight wheel and axle, such that the elevator cab and counterweight are in equilibrium when empty. The elevator cab weighs 1200 kg and is designed to hold 1000 kg. The wheel has a radius of 1.75 m and the axle has a diameter of 2.5 m. All of these components are connected with long lengths of lightweight cables.

   (a) (2 points) If the mass of the counterweight should be less than the mass of the elevator cab, what should the mass of the counterweight be, in kg, and what should the cables connect?



   (b) To test the elevator, the technician sends the elevator up four floors. Shown above is a diagram of the force applied to the elevator cab as a function of time. Positive force acts upwards on the cab.

      i. (3 points) Qualitatively describe the motion of the elevator cab.

      ii. (4 points) What is the speed of the counterweight after 15 s, in $\mathrm{m\,s^{-1}}$?

      iii. (3 points) What is the height of a floor, in meters?

   (c) The elevator is then boarded by three people, each weighing 80 kg, who all want to go up 9 m in 50 s. In order to give the passengers a smoother ride, the function of the force applied to the elevator cab is changed to be

$$F(t) = \alpha \sin(\beta t) + \gamma$$

   where α, β, and γ are constants such that the elevator will only accelerate and decelerate once and that $F_{net}(0) = F_{net}(t_f) = 0$, where $F_{net}(t)$ is the net force on the elevator at time t and t_f is the time at which the elevator reaches a height of 9 m. Find, in their respective SI units:

      i. (4 points) α.

      ii. (3 points) β.

      iii. (2 points) γ.

Solution:

Through this solution, we will use variables $m_1, m_2, T_1, T_2, r_1, r_2, a_1, a_2$ which represent the mass of the elevator cab and the counterweight, the tension in the cable connected to the elevator cab and the counterweight, the radius of the axle and the radius of the wheel, and the acceleration of the elevator cab and the counterweight, respectively.

(a) Since the elevator cab and counterweight are in equilibrium when empty, the forces must balance around the wheel and axle. Also, the elevator cab must connect to the axle and the counterweight must connect to the wheel, so that the latter is less massive than the former. The system is in equilibrium, so $T_1 = m_1 g$ and $T_2 = m_2 g$. Balancing torques and rearranging leads to m_2.

$$T_1 \times r_1 = T_2 \times r_2 \implies m_1 g \times r_1 = m_2 g \times r_2$$

$$m_2 = \frac{m_1 r_1}{r_2} = \frac{1200\,\mathrm{kg} \times 1.25\,\mathrm{m}}{1.75\,\mathrm{m}} = 857\,\mathrm{kg}$$

(b) i. The elevator cab moves upwards with constant acceleration. Then after 10 seconds, it coasts at a constant velocity. Finally after 30 more seconds, it comes to a stop with constant deceleration.

ii. Use $\Sigma F = ma$ on m_1 and m_2, balance torques, and relate accelerations using the conservation of string. With 4 equations and 4 unknowns, we can solve for a_2. We will set the positive direction to be upwards.

$$\begin{cases} m_1 a_1 = T_1 - m_1 g + F \\ m_2 a_2 = T_2 - m_2 g \\ T_1 \times r_1 = T_2 \times r_2 \\ a_1 r_2 = -a_2 r_1 \end{cases} \implies a_2 = \frac{r_1 r_2 (m_1 g - F) - r_2^2 m_2 g}{r_2^2 m_2 + r_1^2 m_1} = -0.035\,\mathrm{m\,s^{-2}}$$

With acceleration, use $s = |at|$ to find final speed.

$$s_{15\,\mathrm{s}} = |-0.035\,\mathrm{m\,s^{-2}} \times 10\,\mathrm{s}| = 0.35\,\mathrm{m\,s^{-1}}$$

This problem can also be solved by balancing the angular momentum around the wheel and axle.

$$\Delta L = F r_1 t = m_1 v_1 r_1 - m_2 v_2 r_2 = -m_1 v_2 \left(r_1 + \frac{r_1^2}{r_2} \right)$$

$$v_2 = \frac{900\,\mathrm{kg\,m^2\,s^{-1}}}{-2570\,\mathrm{kg\,m}} = -0.35\,\mathrm{m\,s^{-1}} \implies s_{15\,\mathrm{s}} = 0.35\,\mathrm{m\,s^{-1}}$$

iii. Use the kinematic equation $\delta x = \frac{1}{2} a t^2 + v_0 t$ over three periods, from 0 s to 10 s, 10 s to 40 s, and 40 s to 50 s.

$$\Delta x_{0\text{-}10} + \Delta x_{10\text{-}40} + \Delta x_{40\text{-}50} = \frac{2.5\,\mathrm{m}}{2} + 7.5\,\mathrm{m} + \frac{-2.5\,\mathrm{m}}{2} + 2.5\,\mathrm{m} = 10\,\mathrm{m}$$

So the height of one floor is 2.5 m.

(c) i. Finding α is the hardest variable out of the three. From the previous system of equations in (b.ii), a_1 can be simplified to

$$a_1 = \frac{F_{net}}{2800\,\text{kg}}$$

Then, with the net force function $F_{net} = \alpha\sin(\beta t)$ and knowing $v(0) = 0$, integrate acceleration twice for position function.

$$\frac{1}{2800}\int_0^{t_f}\int_0^{t'} \alpha\sin(\beta t)dtdt' = \frac{-\alpha}{2800\beta}\int_0^{t_f}\cos(\beta t') - 1dt' = \frac{-\alpha}{2800\beta}\left[\frac{1}{\beta}\sin(\beta t') - t'\right]\Bigg|_0^{t_f}$$

Plugging in for $\beta = 2\pi/t_f\ \text{s}^{-1}$ (solved for below) and $t_f = 50\,\text{s}$,

$$\frac{\alpha}{2800\,\text{kg}\times\beta}\,t_f = 9\,\text{m} \implies \alpha = 63.3\,\text{N}$$

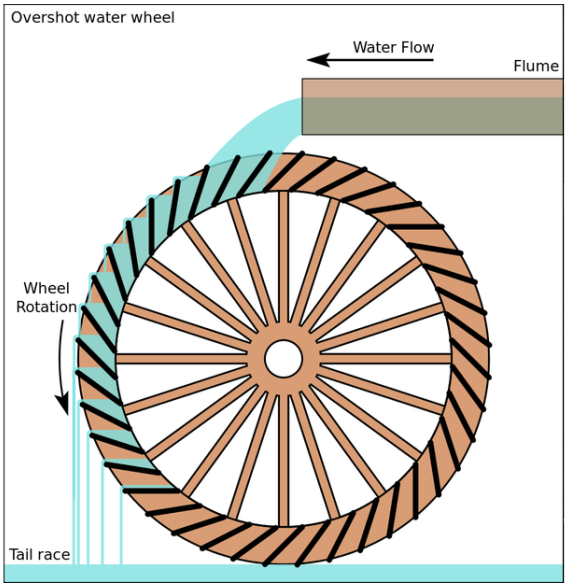
ii. The sine function can only go through one cycle over $50\,\text{s}$, so

$$\beta = \frac{2\pi}{50\,\text{s}} = 0.126\,\text{s}^{-1}$$

iii. The net force on the elevator cab is equal to zero initially, so

$$\gamma = 240\,\text{kg}\times 9.81\,\text{m}\,\text{s}^{-2} = 2350\,\text{N}$$

5. (21 points) A watermill is a mill that uses hydropower.



(a) The diagram on the left shows the waterwheel that powers the mill. The water flows through a flume at a rate of $5.00 \times 10^4\,\mathrm{cm^3\,s^{-1}}$ and falls from a height $6.00\,\mathrm{m}$ above the water level. Recall the density of water is $1\,\mathrm{g\,mL^{-1}}$.

   i. (5 points) Given the waterwheel rotates at 10 rpm, find the maximum torque on the central axle, in N m. Assume the diameter of the wheel is near the height of the flume, the water enters and exits at the top and bottom of the wheel, and the water is evenly distributed between the blades.

   ii. (2 points) If the torque on the central axle is 2250 N m, what is the efficiency of the waterwheel?

   iii. (2 points) Give two possible reasons for the waterwheel's inefficiency.

(b) The diagram on the right depicts the grinding mechanism of the mill. The primary axle rotates at 10 rpm with a torque of 2250 N m, the bevel gears on the primary and the secondary axle have a radius of 60 cm and 1 m, respectively, and the grindstone is a uniform disk of mass 500 kg and radius 1.5 m.

   i. (2 points) The mill owner is considering changing the gears to have helical teeth. Give one advantage and one disadvantage of this change.

   ii. (2 points) Calculate the IMA of the gears.

   iii. (4 points) The coefficient of static and kinetic friction between the grindstone and the stone base is 0.8 and 0.5, respectively. What is the minimum distance it can be set from the secondary axle such that the grindstone rolls without slipping, in meters?

   iv. (4 points) The grindstone is set 0.5 m from the secondary axle. What is the rate at which energy is being lost through friction, in W?

Solution:

(a) i. Under all of the assumptions given in the problem statement, there is conservation of energy, so we can use that to our advantage.

$$P = \tau\omega = Q\rho \times gh \implies \tau = \frac{Q\rho \times gh}{\omega} = \frac{50\,\mathrm{kg\,s^{-1}} \times 9.81\,\mathrm{m\,s^{-2}} \times 6.00\,\mathrm{m}}{\pi/3\ \mathrm{s^{-1}}} = 2810\,\mathrm{N\,m}$$

ii. Using the ideal torque solved for in (a.i), find the efficiency.

$$\eta = \frac{P_o}{P_i} = \frac{\tau_o\omega}{\tau_i\omega} = \frac{2250\,\mathrm{N\,m}}{2810\,\mathrm{N\,m}} = 80\%$$

iii. Water leaks from the blades before it reaches the bottom, waterwheel diameter less than the height of the flume, friction, etc.

(b) i. Advantage: [gears run smoother and quieter]; Disadvantage: [causes a thrust along the axis of the gear, greater sliding friction between the teeth, harder to manufacture/more expensive]

ii.

$$\mathrm{IMA} = \frac{100\,\mathrm{cm}}{60\,\mathrm{cm}} = 1.67$$

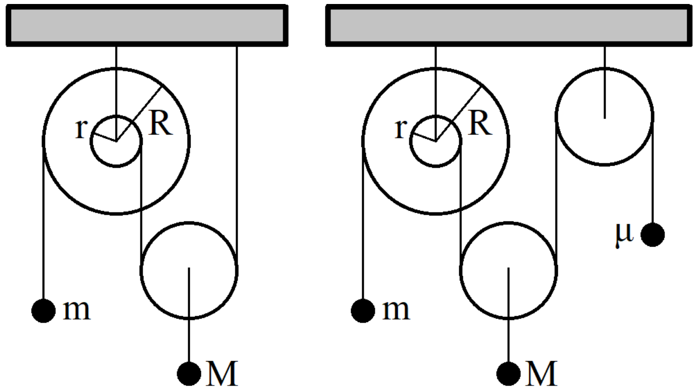
iii. The force on the grindstone cannot exceed the maximum static friction force for the grindstone to roll without slipping. The torque of the secondary axle is multiplied by the IMA in (b.ii).

$$\frac{\tau \times \mathrm{IMA}}{d} \leq mg\mu_s \implies d \geq \frac{\tau \times \mathrm{IMA}}{mg\mu_s} = \frac{3750\,\mathrm{N\,m}}{500\,\mathrm{kg} \times 9.81\,\mathrm{m\,s^{-2}} \times 0.8} = 0.956\,\mathrm{m}$$

iv. Since this the grindstone is set at a distance less than the minimum distance established in (b.iii), the grindstone must be rolling with slipping. Therefore, the energy lost through friction is dependent on the kinetic friction force and the grindstone's velocity.

$$P = mg\mu_k \times \omega r = 500\,\mathrm{kg} \times 9.81\,\mathrm{m\,s^{-2}} \times 0.5 \times \frac{\pi}{5}\ \mathrm{s^{-1}} \times 0.5\,\mathrm{m} = 770\,\mathrm{W}$$

6. (12 points) Shown below on the left is a pulley system of two point masses (m and M) and three pulleys, one movable and two fixed and coaxially connected ($r = 25\,\text{cm}$ and $R = 90\,\text{cm}$). Assume ideal conditions.



(a) (2 points) What is the IMA of the pulley system?

(b) (3 points) If $m = 10\,\text{kg}$ and $M = 75\,\text{kg}$, Calculate the tension in the left and right strings, in N.

The pulley system is modified so that another fixed pulley and a mass μ is added. The new system is shown above on the right.

(c) (2 points) If $m = 50\,\text{kg}$, find M and μ such that the system is in equilibrium, in kg.

(d) (5 points) Find the direction and magnitude of acceleration for m, in $\text{m}\,\text{s}^{-2}$, when $M = 30\,\text{kg}$ and $\mu = 90\,\text{kg}$.

Solution:

(a) The IMA of the system is the product of the individual machines.

$$\text{IMA} = \frac{90\,\text{cm}}{25\,\text{cm}} \times 2 = 7.2$$

The inverse of 7.2, 0.139, was also accepted.

(b) Let T_1 and T_2 be the tension in the left and right strings, respectively. Set up a system of equations using $\Sigma F = ma$ on the two masses, relating the tensions, and the relative acceleration of the masses. We will set up to be positive.

$$\begin{cases} ma_m = T_1 - mg \\ Ma_M = 2T_2 - Mg \\ RT_1 = rT_2 \\ a_m = -\text{IMA} \times a_M \end{cases} \implies T_1 = 102\,\text{N and } T_2 = 366\,\text{N}$$

(c) For this question, mass m is in equilibrium, so $T_1 = mg$ and $T_2 = \text{IMA} \times mg$. Finally, apply $\Sigma F = 0$ to mass M and mass μ.

$$M = \frac{2T_2}{g} = 360\,\text{kg and } \mu = \frac{T_2}{g} = 180\,\text{kg}$$

(d) We have 5 unknowns, a_m, a_M, a_μ, T_1, and T_2, so we need 5 equations to solve for them. We will use $\Sigma F = ma$ on the three masses, balance torques around the pulley, and relate the acceleration of the masses. We will set up to be positive.

$$\begin{cases} ma_m = T_1 - mg \\ Ma_M = 2T_2 - Mg \\ \mu a_\mu = T_2 - \mu g \\ RT_1 = rT_2 \\ a_m = -\frac{R}{r} \times (2a_M + a_\mu) \end{cases} \implies a_m = -8.59\,\text{m}\,\text{s}^{-2}$$

So mass m accelerates downwards at a rate of $8.59\,\text{m}\,\text{s}^{-2}$.

7. (18 points) A 60 lbs wooden door is 80″ tall, 36″ wide, and of negligible thickness. It is designed to close on its own with a constant torque of 12 lbf·ft. Assume the door is currently opened by 30°. Below are the friction coefficients of common materials.

Material	μ_s	μ_k
Wood	0.40	0.25
Plastic	0.75	0.55
Rubber	0.95	0.80

(a) A doorstop, of some angle θ, is placed at the end of the door. Assume the weight of the doorstop is negligible. Using the information provided above, find:

i. (2 points) The magnitude and direction of the normal force on the door from doorstop, in terms of θ and lbf.

ii. (3 points) The maximum possible angle of doorstops of each material that will still hold the door open, in degrees.

iii. (2 points) The IMA of each of those doorstops found in the previous sub-question.

(b) A rubber doorstop with an angle of 10° and a negligible weight to be placed at the end of the door and hold the door open. The door is then opened another 60° and released from rest.

i. (5 points) Calculate the time it takes for the door to hit the doorstop, in seconds.

ii. (4 points) Once the door hits the doorstop, there is a constant normal force of 30 lbf between the door and the doorstop. How much further must the door turn until it comes to rest, in degrees?

iii. (2 points) How much work was done by friction, in N?

Solution:

(a) i. The component of the normal force tangent to the axis of rotation must exert a torque equal to 12 lbf·ft. So

$$N_{\text{door}} \sin\theta \times 3\,\text{ft} = 12\,\text{lbf·ft} \implies N_{\text{door}} = 4\,\text{lbf} \times \csc\theta$$

This force is normal to the plane of the wedge, so it is θ from the normal or $90° - \theta$ from the ground.

ii. Since the weight of the doorstop is negligible, the normal force from the ground is equal to the vertical component of the normal force between the door and the doorstop. The doorstop is also at rest, so we balance the horizontal forces on it, friction and normal force from the door.

$$F_{\text{friction}} = N_{\text{ground}}\mu_s = N_{\text{door}} \cos\theta \times \mu_s = 4\,\text{lbf} \implies \mu_s = \tan\theta$$

$$\theta_{\text{wood}} = 21.8° \text{ and } \theta_{\text{plastic}} = 36.9° \text{ and } \theta_{\text{rubber}} = 43.5°$$

iii. IMA of the doorstop is equal to $\cot\theta$.

$$\text{IMA}_{\text{wood}} = 2.5 \text{ and } \text{IMA}_{\text{plastic}} = 1.33 \text{ and } \text{IMA}_{\text{rubber}} = 1.05$$

(b) i. This is a rotational kinematics question. Using $\tau = I\alpha$, we can find the time it takes to move a certain distance given I and τ. Treating the door as a rod rotating around its end

$$I = \frac{1}{3}mL^2 = \frac{1}{3} \times 60\,\text{lb} \times (3\,\text{ft})^2 = 180\,\text{lb·ft}$$

$$\frac{\tau}{I} = \alpha \text{ and } \Delta\theta = \frac{1}{2}\alpha t^2 \implies 60° = \frac{12\,\text{lbf·ft}}{2 \times 180\,\text{lb·ft}} t^2 \implies t = 5.60\,\text{s}$$

ii. Using conservation of energy, the door comes to a rest when the energy gained through the door closing is equal to the energy lost by the frictional force on the doorstop. For torque, work is equal to $\tau\theta$. Let θ be the angle of the doorstop, ϕ be the additional angle the door will turn until it comes to rest, and d be the width of the door .

$$W_{\text{door}} + W_{\text{friction}} = 0 \implies \tau_{\text{door}} \times (60° + \phi) - N_{\text{door}} \cos\theta \mu_k d \times \phi = 0$$

$$\phi = \frac{\tau_{\text{door}} \times 60°}{N_{\text{door}} \cos\theta \mu_k d - \tau_{\text{door}}} = 12.2°$$

iii. The work done by friction is equal to the work done by the door.

$$W_{\text{friction}} = -\tau_{\text{door}} \times (60° + \phi) = -12\,\text{lbf·ft} \times \frac{1.356\,\text{N m}}{1\,\text{lbf·ft}} \times 82.2° \times \frac{2\pi\,\text{rad}}{360°} = -20.5\,\text{J}$$