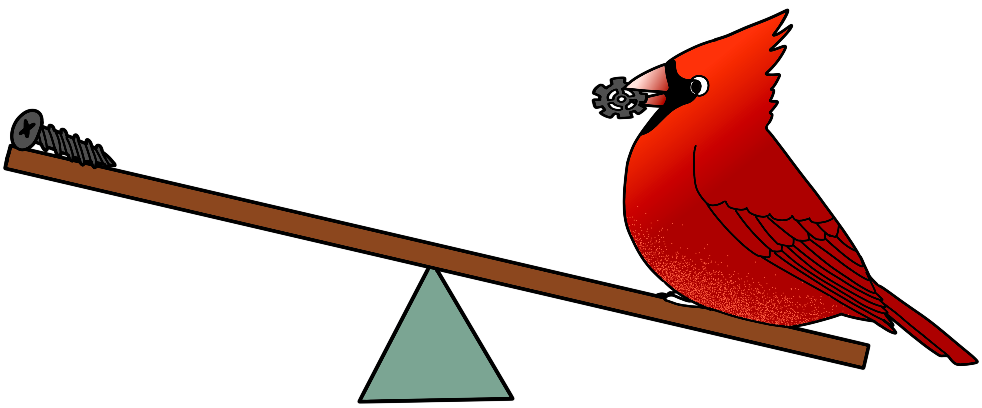
Science Olympiad

Machines C

BirdSO Invitational

March 7-13, 2021



Section B Solutions

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Feedback? **Test Code**: *2021BirdSO-MachinesC-Screw*

Section B: Free Response

Points are shown for each question or sub-question, for a total of 90 points.

1. (10 points) A double started screw is passed through a thin, stiff board such that it is upright. It has a pitch P, a mass m, a shaft radius R, and a cap diameter D. Wrapped around the screw cap is an elastic rope that acts like a spring by applying a force kL at the edge of the cap, where k is a constant and L is the elongation of the rope. Finally, a mass M is placed on the screw cap. Assume no friction between the screw and the board. Answer all sub-questions in terms of the given variables and other fundamental constants.
   (a) (3 points) What is the mechanical advantage of the screw? *(Note: You will have to make an assumption.)*
   (b) (4 points) How many revolutions will the screw make before it reaches equilibrium? Initially L is equal to 0.
   (c) (3 points) The answer to the previous question assumed that the rope's thickness is negligible. Would the number of revolutions increase or decrease if the rope's thickness is considered? Explain. Assume the rope continues to wrap around the screw cap.

Solution:

(a) With the definition of IMA as equal to d_{in}/d_{out}, we consider the rotation of one revolution. Note that the screw is double started.

$$\text{IMA} = \frac{d_{in}}{d_{out}} = \frac{\pi D}{2P}$$

The assumption made is that $m \ll M$, since the weight of the screw could affect the IMA if it was large enough.

(b) We will create two equations: the force applied at the screw cap required to reach equilibrium F_{eq} and the force as a function of the number of revolutions r. Then, we will set them equal to solve for r.

$$F_{eq} = \frac{2P}{\pi D} \cdot (m+M)g \text{ and } F(r) = kL = k \cdot \pi D r$$

$$F(r) = k \cdot \pi D r = \frac{2P}{\pi D} \cdot (m+M)g = F_{eq} \implies r = \frac{2P(m+M)g}{\pi^2 D^2 k}$$

(Note: 0.5 points are deducted for omitting m.)

(c) Decrease, because D increases which increases both the IMA and the rope's elongation (which increases the applied force).

2. (10 points) A chisel is modeled by an isosceles triangular prism with a vertex angle of θ and a mass m. A hammer is modeled by a block of mass M. The chisel is lined up perpendicular to the side of a stone block and hit by the hammer traveling horizontally at velocity V. Subject only to the stone's reaction force, the two are predicted to come to rest after traveling together over a distance D. Answer all sub-questions in terms of the given variables and other fundamental constants.

(a) (2 points) What is the ideal mechanical advantage of the chisel?

(b) (3 points) Determine the average horizontal reaction force from the stone block. Assume the same ideal conditions in the problem statement.

(c) (4 points) The chisel and hammer end up travelling a distance d, where $d < D$. Using the previous force term, find the resultant energy efficiency of the interaction by comparing the initial energy to the work done.

(d) (1 point) Identify a potential source of energy loss other than friction.

Solution:

(a) Since the chisel is a wedge in the shape of an isosceles triangle, the IMA $= \cot(\theta/2)/2$. It is **not** $\cot(\theta)$, which would be the IMA of a right triangle.

(b) This question is comprised of three parts. First, using conservation of momentum, the velocity of the chisel (v) must be calculated from the inelastic collision between the hammer and chisel. Then, find the energy of the new chisel-hammer system (K). Finally, uses the work formula, $W = \bar{F}D$, to find the average force over the distance.

$$MV = (m+M)v \implies v = \frac{MV}{(m+M)} \text{ and } K = \frac{1}{2}(m+M)\left(\frac{MV}{m+M}\right)^2 = \frac{M^2V^2}{2(m+M)}$$

$$W = \bar{F}D = \frac{M^2V^2}{2(m+M)} \implies \bar{F} = \frac{M^2V^2}{2(m+M)D}$$

(Note: 1 point is deducted for omitting the collision step.)

(c) Find the total energy efficiency by considering the efficiency of each step (collision + reaction force) or by comparing the initial energy to the final energy. Half credit for each step. Energy loss from collision is $M/(m+M)$ and from reaction force is d/D. Total efficiency is product of efficiencies of each step:

$$\frac{Md}{(m+M)D}$$

(d) Sound is accepted. Air resistance, wear, and deformation of chisel or wedge are also accepted. Heat is not accepted as it falls under the realm of friction. Deformation of stone is also not accepted as it is where the energy is supposed to go.

3. (25 points) As we cannot conduct the device testing portion of the event, you will draft up a design of a device, which should be able to determine a mass ratio up to 21:2. It **must** consist of an **inclined plane** and a **wheel and axle**, connected together by a **pulley system with an IMA of 2**.

(a) (2 points) Give a one sentence explanation of your device design.

(b) (12 points) Draw **two device diagrams**.

i. Diagram 1:

1. View is from an angle so it is 3-dimensional.
2. Major features are labeled (i.e. simple machine types, mass locations, etc.)

ii. Diagram 2:

1. View is from the side so it is 2-dimensional.
2. Important dimensions are labeled (i.e. simple machine dimensions, device base width, etc.)

(c) (8 points) You are given **3 blocks** with known masses: mass A = 630 g, mass B = 810 g, and mass C = 60 g. You will place two pairs of masses (A&B and A&C, so two masses at a time) on your device such that it is in equilibrium. For each pair of masses:

1. Draw a simple diagram with the location of the masses indicated.
2. Work through the appropriate calculations to show the device is balanced.

You should have **two diagrams** and **two sets of calculations**.

(d) (3 points) Let's assume there is a 5 % error in the angle of the inclined plane. For the second mass pair (A&C), what is the maximum mass ratio (A/C) that could be estimated from your design? Show your work.

Solution:

(a) (2 points) Basic description of the device is given in one sentence.

(b) (6 points) Device consists of the 3 specified machines. (-1 point if pulley doesn't connect the two.)

(1 point) Diagram 1 is isometric.

(2 points) Diagram 1 properly labels major features in the device.

(1 point) Diagram 2 is a side view.

(2 points) Diagram 2 includes dimensions for major features.

(c) For each mass ratio:

(2 points) Diagram with mass locations depicted.

(2 points) Shows proper work with correct calculations to show that the device is in equilibrium.

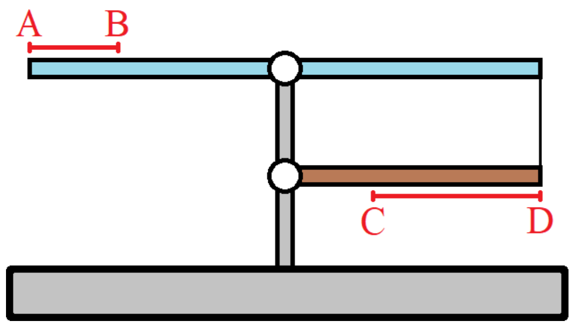
If device does not use the correct machines, points are halved.

(d) For each source of error:

(1 point) Correct maximum mass ratio.

(2 points) Shows proper work.

4. (25 points) In preparation for the device testing portion of the event, Phoebe acquires a (pretty much) massless, frictionless, indestructible lever arm that is 1 m long. However, the morning of the competition, her partner reminds her that the device must be a compound lever system. Panicking, she decides to use a 50 cm long tree branch she finds on the ground as another lever. Shown below is a diagram of her device with the bottom beam being the tree branch. The two beams are connected by a lightweight string. The red ranges AB and CD are 20 cm and 35 cm long. The grey surfaces are immovable surfaces.



(a) (7 points) Let's first assume the mass of the tree branch is negligible and the system is as ideal as can be.

i. (3 points) If Phoebe can only hang masses in the red ranges (defined by points A, B, C, and D), what is the maximum and minimum mass ratios she can determine?

ii. (4 points) If masses only hang in the red ranges, what is the maximum reaction force at the fulcrum of the bottom beam, in N? The device must be in equilibrium. *(Hint: Each mass must be between 20.0 and 800.0 g.)*

(b) (10 points) Unfortunately, since Phoebe has to use the tree branch (which has mass), she needs to collect some information about this non-ideal material. She has two masses X and Y, which are 600 g and 125 g, respectively. She hangs mass X at point A and mass Y at point C and records an instantaneous counterclockwise angular acceleration of $10.4\,\mathrm{rad\,s^{-2}}$. She then hangs mass Y at point B and mass X at point D and records an instantaneous clockwise angular acceleration of $13.7\,\mathrm{rad\,s^{-2}}$. The tree branch can be approximated as a uniform rod.

i. (5 points) What is the frictional torque acting at the fulcrum of the bottom beam, in N m?

ii. (5 points) What is the mass of the tree branch, in kg?

(c) (8 points) Finally, Phoebe needs to figure out the best measurement strategy to get the optimal total score. She estimates that it takes 5 seconds for her to halve the percent error of one mass ratio estimate. *(Note: She must spend the entire 5 seconds each time.)* Assume she starts at 100 percent error for both mass ratio estimates. There are two mass ratios. Recall the formulas for the Time Score (TS) = ((240 − total time spent in seconds) / 240) × 15 points and Ratio Scores (R1 and R2) = (1 − (abs(AR − MR) / AR)) × 15 points, where AR is the actual mass ratio and MR is the estimated mass ratio. The total score is the TS + R1 + R2, making the theoretical maximum total score 45.

i. (4 points) How many seconds should she spend measuring the masses?

ii. (4 points) To three significant figures, what is the expected optimal total score?

Solution:

(a) i. Maximum mass ratio occurs when placing masses at A and C: $50\,\text{cm}/15\,\text{cm} = 3.33$. Minimum mass ratio is up to interpretation. If mass ratio is considered between any two arbitrary masses, it's the reciprocal of the maximum mass ratio: $1/3.33 = 0.3$. But, if mass ratio is considered as larger mass over smaller mass, the minimum would be 1. Both minimum mass ratios accepted, but 0.6 is *not* accepted.

ii. The trick for this question is to consider the torques at point D. Let R_y be the upwards reaction force at the bottom fulcrum, m be the mass on the lower beam, and d be the distance the mass is placed from point D. Since the device is in equilibrium, we know the net torque around any point is 0, so:

$$\Sigma M_D = 0 = R_y \cdot 50\,\text{cm} - mgd \implies R_y = \frac{mgd}{50\,\text{cm}}$$

To maximize R_y, m and d must be at a maximum, giving us:

$$R_y = \frac{0.8\,\text{kg} \cdot 9.81\,\text{m}\,\text{s}^{-2} \cdot 35\,\text{cm}}{50\,\text{cm}} = 5.49\,\text{N}$$

(b) These two questions are answered at the same time by setting up a system of equation from applying $\tau_{net} = I\alpha$ on the two givens. Let the mass of the tree branch be m_r and the frictional torque be τ_f. Note that $I_{point} = md^2$ and $I_{rod} = ml^2/3$. For the first scenario:

$$I_1 = m_X(0.5\,\text{m})^2 + m_Y(0.15\,\text{m})^2 + m_r(0.5\,\text{m})^2/3 \tag{1}$$

$$\tau_{net} = m_X g(0.5\,\text{m}) - m_Y g(0.15\,\text{m}) - m_r g(0.25\,\text{m}) - \tau_f = I_1(10.4\,\text{rad}\,\text{s}^{-2}) \tag{2}$$

For the second scenario:

$$I_2 = m_X(0.5\,\text{m})^2 + m_Y(0.3\,\text{m})^2 + m_r(0.5\,\text{m})^2/3 \tag{3}$$

$$\tau_{net} = m_X g(0.5\,\text{m}) - m_Y g(0.3\,\text{m}) + m_r g(0.25\,\text{m}) - \tau_f = I_2(13.7\,\text{rad}\,\text{s}^{-2}) \tag{4}$$

Solving equations (1)-(4) for m_r and τ_f results in:

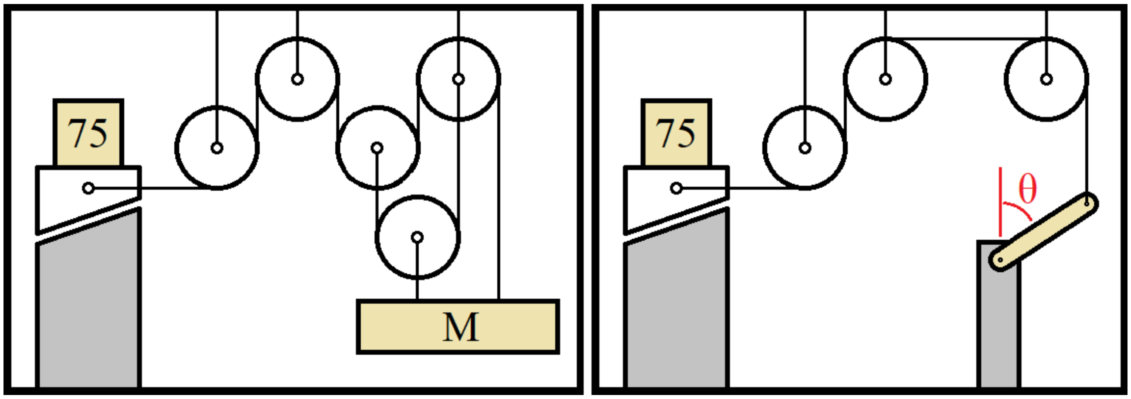
$$m_r = 0.174\,\text{kg} \text{ and } \tau_f = 0.594\,\text{N}\,\text{m}$$

(c) These two questions can also be answered at the same time. Set up a function of the score with respect to time. Make sure to account for both mass ratios, as it take 5 seconds *each* to half the percent error.

$$S(t) = \left(1 - \frac{t}{240}\right) \cdot 15 + \left(1 - 2^{\frac{-t}{10}}\right) \cdot 30$$

Plugging in multiples of 10 for t gives a maximum for $S(t)$ at $t = 50\,\text{s}$, where $S(50) = 40.9$.
(Note: This value is very close to the global maximum of $S(t)$ which is at 50.562 s.)

5. (20 points) Shown below is a diagram of a compound machine system. The inclined plane is at a 15 degree angle, with respect to the horizontal, and a massless, trapezoidal wedge of the same angle is placed on it. The coefficient of static and kinetic friction between the two is 0.15 and 0.10. Fixed on the wedge is a 75 kg mass.



(a) (9 points) The cord is passed through a pulley system and attached to a plate of mass M, shown in the left diagram. The pulleys are massless and frictionless.

i. (2 points) What is the ideal mechanical advantage of the pulley system?

ii. (4 points) What is the maximum and minimum M such that the system is in equilibrium, in kg?

iii. (3 points) Let M be 300 kg. If the plate starts from rest, what is its displacement after 9 seconds, in m? Assume the plate does not hit the ground and the wedge stays on the inclined plane.

(b) (11 points) The cord is now threaded through a simpler pulley system and connected to a rod at an angle θ with respect to the vertical, shown in the right diagram. For simplicity, assume the cord is always tangential to the rod. The rod is 4 meters long.

i. (1 point) Let the applied force on the rod be through the cord. What class lever best describes the rod?

ii. (4 points) What are the maximum and minimum angles (from 0 to 180 degrees) for the system to be in equilibrium, in degrees? Assume a linear density of $15\,\mathrm{kg\,m^{-1}}$.

iii. (6 points) The minimum angle for the system to be in equilibrium can be represented as the function $\theta_{min}(\lambda)$, where λ is the linear density of the rod, in $\mathrm{kg\,m^{-1}}$. Find this function and identify its limits.

Solution:

(a) i. Let the tension in the cord connected to the wedge be T. Balancing forces around the two movable pulleys results in a tension of $2T$ and $4T$ in the other two cords, with the latter connecting to the mass M. A force of $5T$ is applied to mass M, so an IMA of 5 and 1/5 are accepted.

ii. Set up $\Sigma F = ma$ in the parallel and perpendicular directions with respect to the incline. Our system will be the forces acting on the wedge. Notice that $a_{\parallel} = 0$ and $a_{\perp} = 0$ as the system is in equilibrium. Also, let T be the same tension as in (a.i), m be the 75 kg mass, θ be 15°, and α be a number in the range $[-1, 1]$. We will use this T and relate it to mass M in a third equation.

$$\begin{cases} \Sigma F_{\perp} = N - mg\cos\theta - T\sin\theta = 0 \\ \Sigma F_{\parallel} = T\cos\theta - mg\sin\theta - \alpha\mu_s N = 0 \\ 5T = Mg \end{cases} \implies M = 5m \cdot \frac{\sin\theta + \alpha\mu_s\cos\theta}{\cos\theta - \alpha\mu_s\sin\theta}$$

M is at a minimum when $\alpha = -1$ and at a maximum when $\alpha = 1$. Plugging in values, we get:

$$M_{max} = 163\,\mathrm{kg} \text{ and } M_{min} = 42.5\,\mathrm{kg}$$

iii. Similar to (a.ii), we will set up the same first two system of equations along with a $\Sigma F = ma$ on the mass M and a relation between the mass m and mass M accelerations. Four unknowns can be solved by four equations and a constant acceleration equation can be used to find the final displacement.

$$\begin{cases} \Sigma F_{\perp} = N - mg\cos\theta - T\sin\theta = 0 \\ \Sigma F_{\parallel} = T\cos\theta - mg\sin\theta - \mu_k N = ma_m \\ \Sigma F_M = Mg - 5T = Ma_M \\ a_m\cos\theta = 5a_M \end{cases} \implies \begin{cases} a_M = g \cdot \frac{M\gamma - 5m\beta}{M\gamma + 25m\sec\theta} \\ \beta = \sin\theta + \mu_k\cos\theta \\ \gamma = \cos\theta - \mu_k\sin\theta \end{cases}$$

Plugging in values, we get $a_M = 0.656\,\mathrm{m\,s^{-2}}$. Using constant acceleration kinematics, the total displacement of the plate is:

$$\Delta d = \frac{1}{2}a_M t^2 = 26.6\,\mathrm{m}$$

(b) i. The applied force and load (weight) are on the same side of the fulcrum and the applied force is further out than the load, so the lever is a class 2 one.

ii. We can use the minimum and maximum masses we calculated in (a.ii) to determine the minimum and maximum tension T that keeps the wedge in equilibrium. $T_{min} = M_{min}g/5 = 83.4\,\mathrm{N}$ and $T_{max} = M_{max}g/5 = 320\,\mathrm{N}$. This tension T is also applied on the end of the rod and creates a torque. If we let λL be the mass of the rod, we get this equilibrium equation:

$$\lambda Lg\sin\theta \cdot \frac{L}{2} = T \cdot L \implies \sin\theta = \frac{2T}{\lambda Lg} \tag{5}$$

θ_{min} occurs when we use T_{min} and θ_{max} is just the supplementary angle $180° - \theta_{min}$. Plugging in values:

$$\theta_{min} = \sin^{-1}\left(\frac{2 \cdot 83.4\,\mathrm{N}}{15\,\mathrm{kg\,m^{-1}} \cdot 4\,\mathrm{m} \cdot 9.81\,\mathrm{m\,s^{-2}}}\right) = 16.5° \text{ and } \theta_{max} = 180° - \theta_{min} = 163.5°$$

iii. Finally, to come up with the function $\theta_{min}(\lambda)$, we can use equation (5) to get:

$$\theta_{min}(\lambda) = \sin^{-1}\left(\frac{2 \cdot 83.4\,\mathrm{N}}{4\,\mathrm{m} \cdot 9.81\,\mathrm{m\,s^{-2}}} \cdot \lambda^{-1}\right) = \sin^{-1}(4.25\,\mathrm{kg\,m^{-1}} \cdot \lambda^{-1})$$

This function is defined for all large enough λ, as θ_{min} approaches 0 as λ increases. However, there is a lower bound for λ, where it is impossible for the system to reach equilibrium for a small enough λ. This occurs when $4.25\,\mathrm{kg\,m^{-1}} \cdot \lambda^{-1} = 1$, so $\theta_{min}(\lambda)$ is defined for $\lambda \in [4.25, \infty)\mathrm{kg\,m^{-1}}$.