Science Olympiad
Solon Invitational
February 3, 2024

Astronomy C Solutions

Section C Solutions

In this solution guide, references to various sources (e.g. textbooks and online material) are made to guide the reader towards resources to learn the equations more in depth. We hope readers find it useful.

Section C: Quantitative

In this section, you will be asked to perform calculations and provide numerical answers, as well as answer follow up questions that test your understanding. Please box your final answer! Work will not be graded. This section contains a total of 50 points.

Conversions and constants you may find helpful:

$1\,\mathrm{au} = 1.496 \times 10^{11}\,\mathrm{m}$
$1\,\mathrm{ly} = 9.461 \times 10^{15}\,\mathrm{m}$
$1\,\mathrm{pc} = 3.086 \times 10^{16}\,\mathrm{m}$
$1\,\mathrm{M}_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$ (Solar Mass)
$1\,\mathrm{M}_{♃} = 1.898 \times 10^{27}\,\mathrm{kg}$ (Jupiter Mass)
$1\,\mathrm{M}_\oplus = 5.972 \times 10^{24}\,\mathrm{kg}$ (Earth Mass)
$1\,\mathrm{R}_\odot = 6.957 \times 10^{8}\,\mathrm{m}$ (Solar Radius)
$1\,\mathrm{R}_\oplus = 6.371 \times 10^{6}\,\mathrm{m}$ (Earth Radius)
$G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
$b = 2.898 \times 10^{-3}\,\mathrm{m\,K}$
$\sigma = 5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
$M_\odot = 4.74$

Peekaboo! Matilda, a precocious child, is fascinated with astronomy and wishes to discover an exoplanet. She receives a telescope from Miss H and begins observing a star that she dubs Agatha.

1. [2 pts] Agatha has a parallax of 7.24 mas. At what distance d is it located (in parsecs)?

Solution: Using parallax, we can directly calculate d as

$$d = \frac{1}{p} = \frac{1}{7.24 \times 10^{-3}\,\mathrm{arcsec}} = \boxed{138\,\mathrm{pc.}}$$

See §4.1 of *A Student's Guide to the Mathematics of Astronomy* (2013) written by Daniel Fleisch and Julia Kregenow (hereafter referred to as GMA).

2. [2 pts] Matilda determines the apparent magnitude to be +9.15. What is its absolute magnitude?

Solution: Solve for M in the distance modulus equation,

$$m - M = 5\log_{10}\left[\frac{d}{10\,\mathrm{pc}}\right],$$

to find

$$9.15 - M = 5\log_{10}\left[\frac{138\,\mathrm{pc}}{10\,\mathrm{pc}}\right] \implies M = \boxed{+3.45.}$$

See §5.3 of GMA.

3. [3 pts] The effective temperature of Agatha is 8280 K. Calculate its radius $R_\star$ (in $\mathrm{R}_\odot$).

Solution: This question requires two steps. The main equation for determining the radius of star is by using the Stefan–Boltzmann law (§3.2.2 in GMA), which relates luminosity to radius and temperature. However, to find luminosity, we need to convert absolute magnitude into luminosity. This can be done by understanding magnitude represents a ratio of luminosity (or brightness) where 5 magnitude steps correspond to 100 times luminosity (or brightness) (§5.3 in GMA describes this in more detail).

The luminosity–magnitude relation is

$$M - M_\odot = -2.5 \log_{10}\left[\frac{L}{\mathrm{L}_\odot}\right],$$

where $\mathrm{L}_\odot$ represents 1 solar luminosity. With this, our answer from the previous question, and the provided absolute magnitude of the sun, we calculate

$$3.45 - 4.74 = -2.5 \log_{10}\left[\frac{L}{\mathrm{L}_\odot}\right] \implies L = 3.28\,\mathrm{L}_\odot.$$

The Stefan–Boltzmann relation is

$$\frac{L}{\mathrm{L}_\odot} = \left(\frac{R}{\mathrm{R}_\odot}\right)^2 \left(\frac{T}{\mathrm{T}_\odot}\right)^4,$$

where $\mathrm{R}_\odot$ represents 1 solar radius and $\mathrm{T}_\odot = 5772\,\mathrm{K}$ represents the surface temperature of the sun. Plugging in the values we know, we calculate

$$\frac{3.28\,\mathrm{L}_\odot}{\mathrm{L}_\odot} = \left(\frac{R}{\mathrm{R}_\odot}\right)^2 \left(\frac{8280\,\mathrm{K}}{5772\,\mathrm{K}}\right)^4 \implies R = \boxed{0.882\,\mathrm{R}_\odot.}$$

4. [3 pts] Matilda confirms that Agatha is a main sequence star. Using the mass-luminosity relationship for main sequence stars, $L \propto M^{3.5}$, determine the mass of Agatha (in $\mathrm{M}_\odot$).

Solution: Since the given relationship is just a proportionality, we can write it as the equation

$$L = cM^{3.5},$$

where c is an unknown coefficient. We can determine c by finding a point we *know* is true for the given relationship. The simplest one is the sun, which is a main sequence star. Plugging in $L = 1\,\mathrm{L}_\odot$ and $M = 1\,\mathrm{M}_\odot$, we find that $c = 1$ if we use the natural units of solar luminosity and solar mass. This way, we can directly use the luminosity we found in the previous question and solve for M as follows:

$$L = M^{3.5} \implies M = (3.28\,\mathrm{L}_\odot)^{1/3.5} = \boxed{1.40\,\mathrm{M}_\odot.}$$

Oftentimes in astronomy, proportionality statements arise from physical or heuristic arguments. These relationships must be fixed to one (or more) known "examples" where they apply before they can be used quantitatively. However, they are useful on their own for relative arguments. E.g. Solar-mass stars A and B have a luminosity ratio of 16:1. From Stefan–Boltzmann, we know they have a 4:1 radius ratio.

One night, Matilda has a dream about Agatha. Floating in the air on a hot air balloon, she finds herself on a gas giant orbiting Agatha. She charts the sky and runs into fantastical beasts zipping through the planet's atmosphere. Most importantly, she determines the planet's circumference is 483 000 km and its orbit period is 56.4 years. She wakes up from her long dream and makes some calculations. (If you were unable to determine either the radius or mass of the star in Questions 3 or 4, you may substitute values of $1\,\mathrm{R}_\odot$ and $0.5\,\mathrm{M}_\odot$. Do <u>not</u> substitute only one value.)

5. [2 pts] Estimate the transit depth δ for the gas giant.

Solution: These next four questions follow a "reversed" transit problem. This is because we start with the planet's radius (or circumference in this case) and predict the transit we expect to see. *Transiting Exoplanets* (2010) by Carole A. Haswell (hereafter TE) is a wonderful introduction to transits.

Using the circumference formula, we determine the radius of the planet to be:

$$r = \frac{P}{2\pi} = \frac{4.83 \times 10^8\,\mathrm{m}}{2\pi} \times \frac{1\,\mathrm{R}_\odot}{6.957 \times 10^8\,\mathrm{m}} = 0.110\,\mathrm{R}_\odot.$$

Then, we use our result from Question 3 and Equation (1.18) in §1.4 of TE to determine the transit depth:

$$\frac{\Delta F}{F} = \left(\frac{0.110\,\mathrm{R}_\odot}{0.882\,\mathrm{R}_\odot}\right)^2 = \boxed{0.0157.}$$

6. [4 pts] For simplicity, Matilda assumes the planet follows a circular orbit that is viewed edge-on. She also assumes $d \gg a \gg R_\star \gg R_\mathrm{p}$, where a and R_p are the planet's semi-major axis and radius, respectively. If she were to observe a transit, estimate the transit duration (in hours). Note that the transit duration is the time between first and fourth contact. (If you were unable to determine the radius of the star in Question 3, you may substitute a value of $1\,\mathrm{R}_\odot$.)

Solution: To answer this question, §3.1 of TE provides the necessary background information. Using Kepler's third law, we can find the semi-major axis, a, of the planet's orbit:

$$\frac{a^3}{P^2} = \frac{G(M_\star + M_\mathrm{p})}{4\pi^2} \approx \frac{GM_\star}{4\pi^2} \implies a = 2.46 \times 10^{12}\,\mathrm{m}.$$

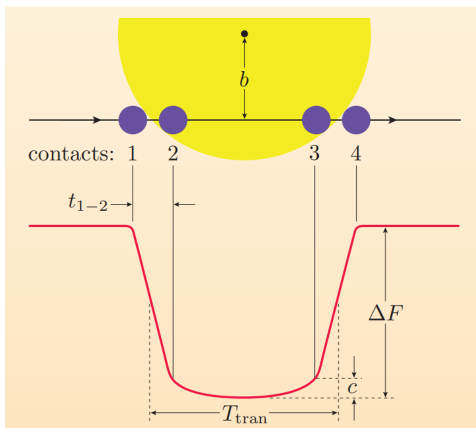
With a, we can find the transit duration using Equation (3.4) in TE to be (with $i = 90°$):

$$\begin{aligned} T_\mathrm{dur} &= \frac{P}{\pi}\sin^{-1}\left(\frac{R_\star + R_\mathrm{p}}{a}\right) \\ &= \frac{56.4\,\mathrm{yr} \times (3.16 \times 10^7\,\mathrm{s\,yr^{-1}})}{\pi}\sin^{-1}\left(\frac{(0.110\,\mathrm{R}_\odot + 0.882\,\mathrm{R}_\odot) \times (6.957 \times 10^8\,\mathrm{m\,R_\odot}^{-1})}{2.46 \times 10^{12}\,\mathrm{m}}\right) \times \frac{1\,\mathrm{h}}{3600\,\mathrm{s}} \\ &= \boxed{44.1\,\mathrm{h}.} \end{aligned}$$

7. [4 pts] Using the values found in the last two questions, plot the transit light curve she expects to see. Indicate the transit depth and the time between each of the four contacts (in hours). Assume there is no limb darkening and that the flux changes linearly.

Solution: The following figure (Figure 3.17 in TE) pretty much encapsulates the light curve we were looking for. We wanted clear indication of axes (not depicted in figure) with labels, tick marks, and units. We also wanted the transit depth clearly labeled and the times between all four consecutive contacts. This can be calculated by modifying Equation (3.4) as

$$t_{1-2} = t_{3-4} = \frac{P}{\pi} \sin^{-1}\left(\frac{R_p}{a}\right),$$
$$t_{2-3} = \frac{P}{\pi} \sin^{-1}\left(\frac{R_\star - R_p}{a}\right).$$



8. [3 pts] Ultimately, the likelihood of Matilda detecting the gas giant with the transit method depends on the orientation of the orbital plane relative to her. It can be shown that the geometric transit probability can be approximated as $R_\star/a$. This equation suggests that transits are most probable for what type of planetary systems?

Solution: Using the given probability, $R_\star/a$, it can be seen that if the host star is larger or if the semi-major axis is smaller, then the overall expression and thereby the transit probability is greater.

This makes intuitive sense because if the host star is larger, there is a wider range of planet inclinations that still result in a transit. And if the semi-major axis is smaller, the range of inclinations that result in a transit also increases. Even though the transit method has been shown to be extremely powerful, especially with large swaths of distributed ground- and space-based telescopes, there are limitations and biases that need to be considered!

Due to the low likelihood of the orbit plane aligning and the long orbit period of the planet, Matilda isn't able to detect it and decides to search for a planet at a different star.

Alien Woes

You are an alien astronomer trying to detect an exoplanet around the Sun. In this question, we'll investigate the challenges with detecting the largest and brightest planet in the Solar System, Jupiter.

The most direct method of detection is, well, direct imaging. The two key parameters that make it difficult are the distance you are from the Solar System and the contrast ratio between the Sun and Jupiter. Let's be generous and suppose you live on Proxima Centauri b, the closest exoplanet to the Solar System, which is located 4.2 light years from the Solar System.

9. [2 pts] Knowing that Jupiter orbits at a distance of 5.2 au from the Sun, what is their angular separation (in arcseconds) when viewed from Proxima Centauri b?

Solution: These six questions about direct imaging are heavily inspired by §1.1 of TE.

This first question can be answered by understanding angular size (§4.2 in GMA):

$$\frac{\text{physical size}}{\text{distance}} = \text{angular size (rad)}$$
$$\implies \frac{5.2\,\text{au}}{4.2\,\text{ly} \times (6.324 \times 10^4\,\text{au}\,\text{ly}^{-1})} = 1.96 \times 10^{-5}\,\text{rad} \times \frac{(3600 \times 180)''}{\pi\,\text{rad}} = \boxed{4.04''.}$$

10. [2 pts] The majority of the light from Jupiter comes from reflected sunlight. Compute the (bolometric) luminosity of Jupiter (in $L_\odot$), which has a radius of 71 500 km, if it reflects 70 % of the sunlight casted onto it.

Solution: Since the "majority of the light from Jupiter comes from reflected sunlight", we take that to mean the bolometric luminosity of Jupiter is equal to the amount of sunlight reflected. To determine the amount of sunlight reaching Jupiter, we imagine a sphere of radius 5.2 au around the Sun. The amount of sunlight that hits Jupiter can be approximated as a circle with a radius of 71 500 km. By dividing the area of this circle by the surface area of the sphere, we can determine the ratio of luminosity hitting Jupiter of which we take 70 %. This process is calculated out as

$$(1\,L_\odot) \times \frac{\pi R_J^2}{4\pi d_{\odot\to J}^2} \times (70\,\%) = \boxed{1.48 \times 10^{-9}\,L_\odot.}$$

As can be seen by the values you calculated, it will be extremely difficult for you to detect Jupiter directly as both the angular separation and contrast ratio are low. However, there is one way you could improve the contrast ratio: observe in the infrared. Image 18* shows the spectral energy distribution (SED) of the Sun and a few planets.

11. [2 pts] Notice that the SED is a log-log graph. Why wouldn't a linear or semi-log graph be well suited to representing this data?

Solution: There is a large range of magnitudes large range of magnitudes in both the x-axis (wavelength) and y-axis (intensity). If a large range of magnitudes linear scale were used for either axis, it would be difficult to recognize the key features in the plot.

12. [2 pts] There are two peaks in the SED for all of the planets. Explain what each peak corresponds to.

Solution: The peak at 0.5 μm corresponds to the reflected sunlight and the peak at 9-20 μm corresponds to thermal emission.

13. [2 pts] Use Wien's law to estimate the effective surface temperature of Mars (in °C).

Solution: Using what we found in the last question, we know that the peak at the longer wavelength corresponds to the effective surface temperature of the planet. Looking at the SED of Mars, the second peak is somewhere in the range of 10-13 μm. Using Wien's law (see §3.2 of GMA), we get:

$$T_{\text{eff}} = \frac{b}{\lambda_{\text{peak}}} \implies \begin{cases} \frac{2898\,\mu\text{m K}}{10\,\mu\text{m}} - 273\,\text{K} & = \boxed{14.8\,^\circ\text{C},} \\ \frac{2898\,\mu\text{m K}}{13\,\mu\text{m}} - 273\,\text{K} & = \boxed{-51.6\,^\circ\text{C}.} \end{cases}$$

14. [2 pts] Over the range of wavelengths depicted, which wavelength gives the best contrast ratio? Determine this contrast ratio to the nearest magnitude.

Solution: By visual inspection, 100 μm gives the best contrast ratio of 10^{-4}.

This set of questions goes to show the limitations of and considerations needed for directing imaging of exoplanets.

*Carole A. Haswell, *Transiting Exoplanets*, 22

Alien Woes cont.

With this information, you draft and submit a proposal to build a space-based telescope that can observe the Solar System at the wavelength determined in the previous question. The review board of the National Alien Space Administration (NASA) brings up some issues that were not addressed in your proposal.

15. [4 pts] Infrared telescopes are a challenging engineering problem due to the many sources of noise affecting their measurements. The board would like (1) an explanation of the primary source of noise and (2) two methods that could be employed to mitigate it. *(Hint: JWST suffers from the same challenges.)*

Solution: Mirrors at normal (room) temperatures emit infrared radiation of their own. This contributes an undesirable thermal noise to the measurements of the telescope. The James Webb Space Telescope (JWST) and other infrared satellites (like Spitzer) had to come up with novel engineering solutions to address this. JWST's approach to cooling its mirrors was to place it at the L2 lagrange point—a point that sits along the imaginary line connecting the Earth and the Sun. This point allows the Earth to shade the telescope from the Sun and reduce the incident solar radiation. To further cool the mirrors, JWST also used multiple layers of heat shields that deflect any residual radiation (likely from the thermal emission of Earth itself) away.

On the other hand, Spitzer—launched two decades prior—placed itself on a heliocentric orbit (around the Sun) where it slowly trailed and drifted away from Earth's orbit. It was placed on this orbit, as opposed to a low-Earth orbit, for a variety of reasons: (1) it limits the heat received from Earth, (2) it allows for greater thermal stability with the absence of eclipses by the Earth, and (3) it takes it outside the Van Allen radiation belts which are full of charged particles that would pelt the spacecraft. On this heliocentric orbit, it cryogenically cooled its telescope using liquid helium and passive radiators.

All these methods have their benefits and drawbacks and are one of the many things aerospace engineers must consider when designing a space telescope.

16. [3 pts] The choice of wavelength is a multifaceted issue and should not be solely based on its contrast ratio. The board would like the proposal to identify an advantage of observing at 20 μm over the wavelength in the original proposal.

Solution: It can be seen that the flux from Jupiter is over 20 times greater at 20 μm as opposed to 100 μm. It may be more beneficial to measure at a wavelength that emits more flux when weighed against the cost of a reduced contrast ratio. It would be fruitless to observe at a wavelength that emits flux below the noise floor of the telescope, as it would then be impossible to detect it at all!

You submit your revised proposal and after ten months the review board gets back to you. "Direct imaging seems like a complete hassle. And (more importantly) building the telescope is too expensive. Try again." Determined to be the first to detect Jupiter, you consider applying the radial velocity method. For simplicity, let's assume Jupiter orbits in a perfect circle and it takes 11.9 years for it to complete one revolution.

17. [2 pts] How fast does Jupiter orbit (in $\mathrm{km\,s^{-1}}$)?

Solution: From Question 9, we know that the orbital separation of Jupiter from the Sun is 5.2 au. Since Jupiter is much larger than the Sun (around 1000 times), we can assume this separation is pretty close to the radius of Jupiter's orbit. Using the circumference formula to find the distance Jupiter travels in one revolution and dividing it by its duration, we get:

$$v_{♃} = \frac{d}{t} = \frac{(5.2\,\mathrm{au}) \times (1.496 \times 10^8\,\mathrm{km\,au^{-1}}) \times 2\pi}{(11.9\,\mathrm{yr}) \times (365.25\,\mathrm{d\,yr^{-1}}) \times (86\,400\,\mathrm{s\,d^{-1}})} = \boxed{13.0\,\mathrm{km\,s^{-1}}.}$$

18. [2 pts] If the Sun has a mass of $1000\,\mathrm{M}_{♃}$, how much would the Sun wobble (in $\mathrm{m\,s^{-1}}$)?

Solution: Binary systems have a useful ratio relationship:

$$\frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{M_2}{M_1}.$$

Since we know the ratio of the mass of the Sun to that of Jupiter is 1000, the Sun must orbit at a speed 1000 times lower, which is $\boxed{13.0\,\mathrm{m\,s^{-1}}.}$

19. [2 pts] The value found in the previous question is the "best" case in observing the Sun. Why?

Solution: The maximum velocity that can be observed is affected by the inclination of their orbit relative to Proxima Centauri b. If the orbit plane is inclined such that it is viewed edge on, then the inclination $i = 90°$ and the Sun's wobble can be detected by the doppler effect. However, if the orbit plane is inclined such that it is viewed face on, then $i = 0°$ and there is no radial movement to detect. This is characterized by a $\sin i$ term, where if v is the velocity of the Sun's wobble, then the velocity that can be observed on Proxima Centauri b is $v \sin i$.

20. [2 pts] Spectral resolution is defined as $R = \lambda/\Delta\lambda$ where $\Delta\lambda$ is the smallest difference in wavelengths that can be distinguished at a wavelength of λ. Scientists on Proxima Centauri b have developed high-resolving-power spectrographs with a spectral resolution of $R \sim 150\,000$. Would it be possible to resolve the wobble of the Sun with two measurements? Justify your answer by finding the minimum velocity that can be resolved (in meter/s).

Solution: Using the given equation and the low-speed redshift equation ($\Delta\lambda/\lambda = v/c$, see §3.3 of GMA), we can calculate the minimum velocity that can be resolved to be:

$$v = \frac{c}{R} = \frac{3 \times 10^8\,\mathrm{m\,s^{-1}}}{150\,000} = \boxed{2000\,\mathrm{m\,s^{-1}}.}$$

This is much greater than the Sun's wobble, so it cannot be resolved.

This investigation gives you a lot to think about. You don't feel like writing another proposal so you stash your findings away and leave it for another day.