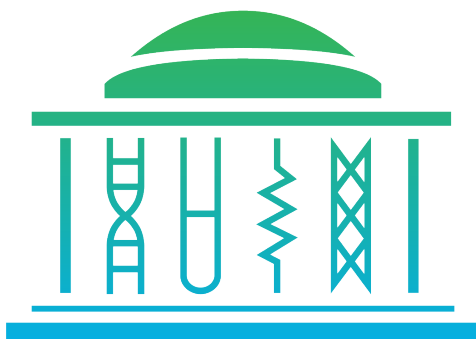


Science Olympiad  
MIT Invitational

January 24, 2026

# Machines C Walkthrough

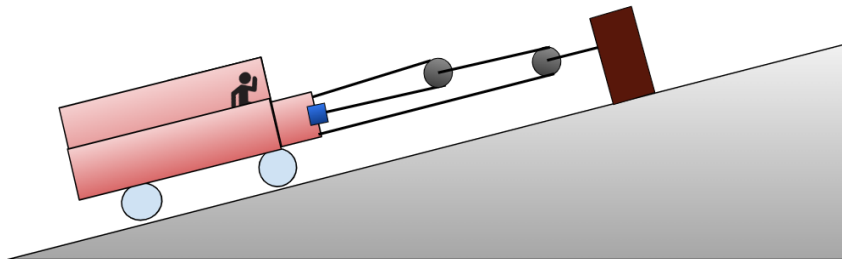


In this walkthrough, we will go over the questions in Section B: Free Response. We hope readers find it useful. Emir Naduvil wrote questions 1–3, with the help of Aneesh Iyer for question 2, and Robert Lee wrote questions 4–5.

## Section B: Free Response

There are 5 multi-part questions in this section, for a total of 80 points.

1. **Aneesh's Automobile Adventure.** Aneesh's 2200 kg truck gets stuck in the cornfields of Indiana as he travels up a  $12^\circ$  degree incline. He uses an electric truck winch motor with nylon cables (originating from the middle cable segment on the truck at the blue box) and links this system to an upright log in the distance. Both the cable and pulley blocks (the black circles) have negligible mass, and the total mass of the truck includes the mass of Aneesh.



- (a) [2 pts] Determine the IMA of the winch system.

**Solution:** Consider the left three rope segments that are linked to the truck. Each of these has tension  $T$  since it is directly attached to the car. Looking at the leftmost winch drum, to maintain equilibrium, the rope segment connecting the two drums and opposite the top two strings must have tension  $T + T = 2T$ . Looking at the rightmost winch drum, we have a  $2T$  pull from the top of the drum, so we must have a  $2T$  pull from the bottom of the drum. Hence, our overall tension on the other side of the drum is  $2T + 2T = 4T$ , and our resulting IMA is hence  $\boxed{4}$ .

This is an example of the [Spanish Burton](#) rig system which is widely useful in rescue operations.

- (b) [3 pts] The cable is well-worn and frayed, so the maximum total tension of any cable segment is 2000 N. Find the minimum static friction coefficient  $\mu_s$  such that the winch can support the truck without snapping the cable.

**Solution:** The maximum tension force,  $T_{\max}$ , for any given rope segment is  $T_{\max} = 2000$  N; since the top rope has tension  $4T$ , which is the overall tension, that is our maximum. Because the tension force is pulling the car up the ramp, the car's gravitational force pulls it down the ramp, so the frictional force must be acting in the same direction as our tension force. Equating our forces, we have that

$$T_{\max} + mg\mu_{\min} \cos(\theta) = mg \sin(\theta).$$

Plugging in our known  $m = 2200$  kg,  $g = 9.81$  m/s<sup>2</sup>,  $\theta = 12^\circ$ , and  $T_{\max} = 2000$  N, we have that

$$2000 + 2200 \cdot 9.81 \cdot \mu_{\min} \cos(12^\circ) = 2200 \cdot 9.81 \cdot \sin(12^\circ) \quad \Rightarrow \quad \mu_{\min} \approx \boxed{0.118}.$$

Answers within  $\pm 0.006$  are accepted.

The reason why friction acts upward is that the object has a tendency to move down, so friction opposes that by moving up.

- (c) [3 pts] Fortunately, Aneesh has some supplies to reinforce the cable and now it can withstand anything. He powers the winch and accelerates up the incline. As he passes through a muddy patch with a kinetic friction coefficient  $\mu_k = 0.18$  at 1 m/s, the truck accelerates at  $0.25 \text{ m/s}^2$ . Compute the tension of the cable (in N) originating from the winch at that instant.

**Solution:** Using our  $F_{\text{net}}$  equation, we have a friction (because it resists the sliding gravitational force down) and tension force (quadrupled from IMA) acting upward and gravity + frictional forces acting downwards. We have that

$$\begin{aligned} F_{\text{net}} &= T_{\text{total}} - F_{g,\parallel} - F_f, \\ T_{\text{total}} &= F_{\text{net}} + F_{g,\parallel} + F_f \\ &= 2200 \cdot 0.25 + 2200 \cdot 9.81 \cdot \sin(12^\circ) + 2200 \cdot 9.81 \cdot \cos(12^\circ) \cdot 0.18 \approx 8840 \text{ N}. \end{aligned}$$

Then, to find the tension in the original center cable, we divide by 4, so we get

$$T_{\text{center}} = \frac{8840 \text{ N}}{4} = \boxed{2210 \text{ N.}}$$

Answers within  $\pm 60 \text{ N}$  are accepted. Friction acts downward since the car is moving up.

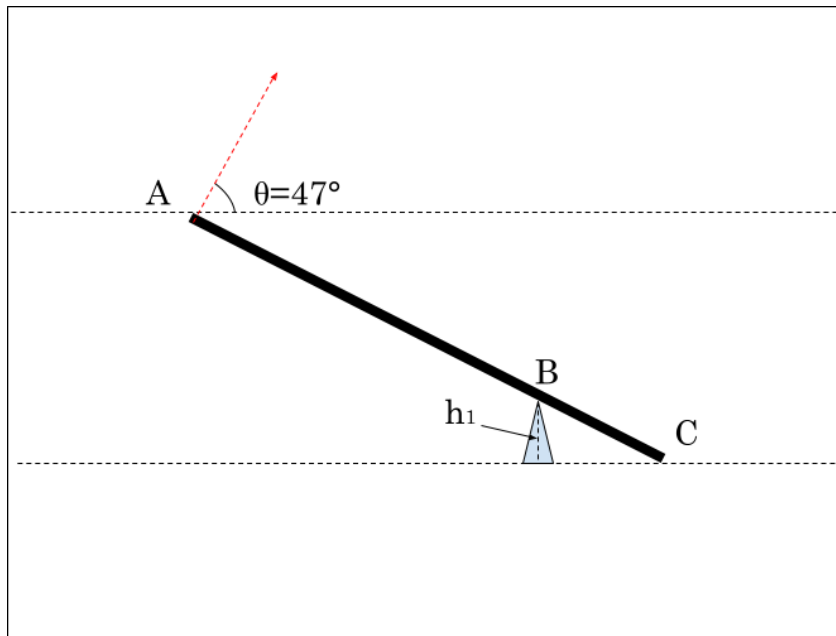
- (d) [2 pts] If the IMA in part (a) is increased, would the instantaneous power consumption of the winch motor in part (c) be greater, lesser, or the same? Justify.

**Solution:** The power consumption would be the same (+1). The car's velocity is fixed in part (c), and since the overall tension to uphold the car must be the same, the power is merely determined by  $P = F \cdot v$ , which is not changed by IMA (+1). In general, increasing IMA would decrease the tension force but require the cable velocity of the winch to increase at a proportional amount, so the product of the two values would still give the original power.

2. **LeBron's Levers.** Our glorious King James needs your help to make an impossible shot! He attempts to launch a basketball across the entire court using machines. A class one lever of length 4 m and mass 1 kg is placed so that the fulcrum of height 50 cm is a distance 30 m away from the hoop. A ball of mass 600 g is placed on the far side of the lever (going further away from the target hoop), and a mass 2 kg is suspended a distance  $h_2$  m above the other end of the lever. You may assume that the hoop is exactly 3 meters tall, and that both masses are point masses located exactly at the ends of the lever.

- (a) [3 pts] To achieve a launch angle  $47^\circ$ , what is the required IMA of the lever? You may assume that the ball will leave the lever at an angle perpendicular to the plane at its final point of motion.

**Solution:**



With a launch angle  $\theta = 47^\circ$ , it can be shown through that the lever will be inclined at an angle  $90^\circ - \theta = 43^\circ$ . For the effort arm, the opposite of this angle will be  $h_1$  (the dotted height of the fulcrum) and the hypotenuse will be the length of the effort arm, giving a length of

$$\frac{h_1}{\sin(90^\circ - \theta)} = \frac{h_1}{\cos(\theta)}.$$

The length of the corresponding load arm is this length subtracted from the total, or  $L - h_1/\cos(\theta)$ . Finding IMA and plugging in our values, we get a final IMA of

$$\frac{BC}{AB} = \frac{\frac{h_1}{\cos(\theta)}}{L - \frac{h_1}{\cos(\theta)}} = \frac{h_1}{L \cos(\theta) - h_1} = \boxed{0.224}.$$

Answers within  $\pm 0.003$  are accepted.

- (b) [3 pts] Once the ball is launched, what is the required initial velocity  $v_0$  (in m/s) in order for it to reach the hoop? You may assume that the ball does not accelerate after leaving the lever.

**Solution:** The initial horizontal distance that the ball is away from the hoop is equal to the 30 m distance of the fulcrum + the distance away from the fulcrum that the ball is. This is equal to

$$30 + 3.266 \cdot \sin(\theta) = 30 + 3.266 \cdot \sin(47^\circ) = 32.388 \text{ m.}$$

Using kinematics yields the following for the horizontal components,

$$32.388 = v_0 \cos(47^\circ)t$$

and this equation for the vertical components

$$3 = 2.728 + v_0 \sin(47^\circ)t - \frac{9.81t^2}{2}.$$

One can find  $v_0 t = 32.388 / \cos(47^\circ)$ . Plugging this into the second equation, we isolate  $t$  in the first equation and plug numbers into the second equation, we get an equation solely in terms of  $t$ :

$$3 = 2.728 + 32.388 \cdot \frac{\sin(47^\circ)}{\cos(47^\circ)} - \frac{9.81t^2}{2}.$$

Solving for  $t$  yields us  $t = 2.651 \text{ s}$ . Plugging this back into  $v_0 t = 32.388 / \cos(47^\circ)$  yields

$$v_0 \approx 17.9 \text{ m/s.}$$

Answers within  $\pm 0.5 \text{ m/s}$  are accepted.

- (c) [4 pts] What is the required energy  $E_1$  (in J) that must be applied to the near side of the lever to achieve this final velocity? Note that from the parallel axis theorem on any rigid body we have that  $I = I_{\text{cm}} + md^2$ , where  $I_{\text{cm}} = 1/12 \cdot mL^2$  is the moment of inertia for the rod about the center of mass,  $m$  is the mass of the object,  $d$  is the perpendicular distance between the new axis and the axis through the center of mass.

**Solution:** This problem is slightly more complicated than initially assumed, as the changing center of mass of the lever must also be considered when applying energy conservation throughout the rotation. To find the final kinetic energy of the lever, we must determine both the final angular velocity and the moment of inertia of the lever.

Based on **part (b)**, we know the initial linear velocity of the ball, which corresponds to the final linear velocity of the lever. The lever arm corresponding to this velocity has a length of

$$l_{\text{arm}} = 4 - \frac{0.5}{\cos(47^\circ)} = 3.267 \text{ m.}$$

This means that the angular velocity of the lever at that moment was

$$\omega = \frac{v}{r} = \frac{17.866}{3.267} = 5.469 \text{ rad/s.}$$

To calculate the moment of inertia of the lever, we can use the parallel axis theorem (where  $x$  represents the distance of the pivot from one side of the lever):

$$I = \frac{1}{12}mL^2 + m(x - L/2)^2 = \frac{1}{12}(1)(4)^2 + (1)(0.733 - 2)^2 = 2.94 \text{ kg} \cdot \text{m}^2.$$

Using geometry, it can be found that the initial height of the center of mass of the lever is  $2h_1/3.267 = 0.306 \text{ m}$ , and that the final height is  $2 \cdot \sin(47^\circ) = 1.463 \text{ m}$ . Knowing the final height of the mass to be  $2.728 \text{ m}$  from **part (b)**, we can plug these values into the law of energy conservation:

$$\begin{aligned} E_0 &= E, \\ E_1 + U_{\text{lever},0} &= K_{\text{rot}} + U_{\text{lever}} + U_{\text{mass}}, \\ E_1 + (1)(9.81)(0.306) &= \frac{1}{2}(2.94)(5.469)^2 + (1)(9.81)(1.463) + (0.6)(9.81)(2.728), \\ E_1 + 3.00 &= 43.97 + 14.34 + 17.20, \\ E_1 &= \boxed{71.3 \text{ J.}} \end{aligned}$$

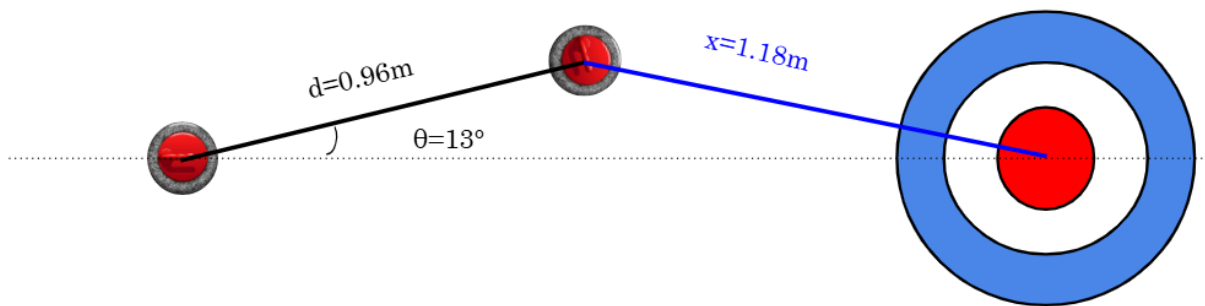
Answers disregarding the center of mass of the lever may receive three points, getting an answer of

$$\boxed{E_1 = 57.0 \text{ J.}}$$

Answers within  $\pm 0.7 \text{ J}$  are accepted.

**Huge shoutout to Aneesh Iyer for this problem!**

3. **Shuffled Scuffboard.** Consider the mini-curling setup below.



This is a portion of a much larger, smooth curling board surface. The target has three regions; the blue and white strips (worth 2 and 4 points, respectively) have thickness 8 cm and the 'bullseye' red circle (worth 7 points) has radius 6 cm. Each mini-curling puck has radius 2.8 cm and mass 45 g. A puck must be fully in a region for points to be scored; if it is in two regions, the lower score is taken. Assume all collisions are perfectly elastic.

(a) Answer some questions about the mechanics of this setup:

- i. [1 pt] To launch the puck, you use your bicep to push it. What class lever are you using?

**Solution:** The human bicep has fulcrum at the elbow, hand as the load, and the muscle effort in between both - making it a **class III** lever.

- ii. [2 pts] Your bicep has  $\text{IMA} = 0.11$ . The curling stone has a handle of length 4.5 cm and a resistive band of length 4 cm. What is the total IMA of the arm-band system?

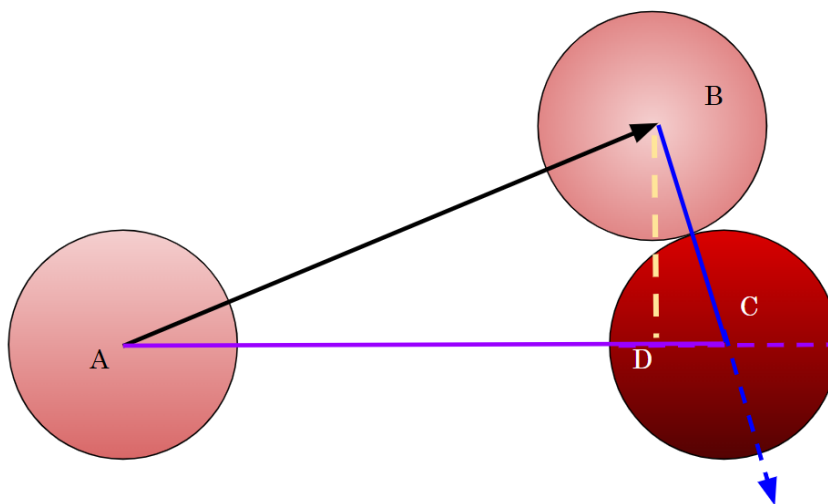
**Solution:**

$$\text{IMA}_{\text{handle}} = \frac{d_{\text{in}}}{d_{\text{out}}} = \frac{4.5 \text{ cm}}{4 \text{ cm}} = 1.125$$

$$\text{IMA}_{\text{total}} = \text{IMA}_{\text{handle}} \cdot \text{IMA}_{\text{arm}} = 1.125 \cdot 0.11 = \boxed{0.124}$$

- (b) [3 pts] You attempt to hit the second puck, which is  $13^\circ$  north relative to the horizontal and  $0.96\text{ m}$  away. You want to find the range of angle(s) such that you still hit the puck. Find the minimum such angle (relative to the horizontal).

**Solution:**



Let the center of the first puck be A, the new center of the first puck upon collision be B, and the center of the second puck be C. Note that we make the  $13^\circ$  difference horizontal for calculation purposes, which we will then add back on later.

The maximum angular displacement from the horizontal such that the balls still hit happens when the balls just touch at a point of tangency such that the velocity vector of the first puck (the black line) and the second puck (the blue line) are exactly perpendicular (convince yourself this is true!). This means we can form a right triangle  $\triangle ABC$  at this minimum.  $\angle BAC$  can be computed using tangency as  $\sin \angle BAC = BC/AC$ . We can find  $BC = 2 \cdot r_{\text{puck}} = 2 \cdot 2.8\text{ cm} = 5.6\text{ cm}$ , and we are given that the distance between the balls, AC, is  $AC = 0.96\text{ m} = 96\text{ cm}$ . Then, we have

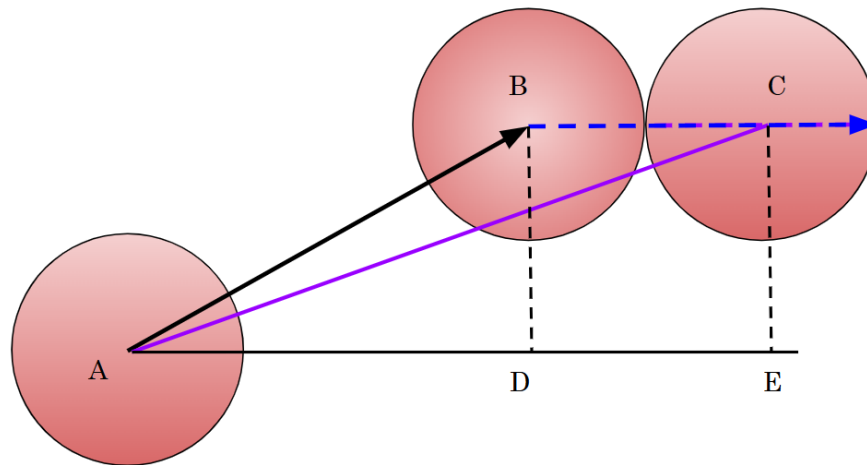
$$\sin \angle BAC = \frac{BC}{AC} = \frac{5.6\text{ cm}}{96\text{ cm}} \implies \angle BAC \approx 3.34^\circ.$$

From here, we factor in the  $13^\circ$  shift. To find the minimum, we consider  $\angle BAC$  to be below the horizontal, which means we subtract  $\angle BAC$  instead of adding it. This gives us a final answer of  $\theta = 13^\circ - \angle BAC = 13^\circ - 3.34^\circ = \boxed{9.66^\circ}$  as our minimum angle. Answers within  $\pm 0.19^\circ$  are accepted.



- (c) [3 pts] You hit the puck such that the resultant velocity of the second puck is strictly horizontal. At what angle must you launch the first puck such that this happens?

**Solution:**

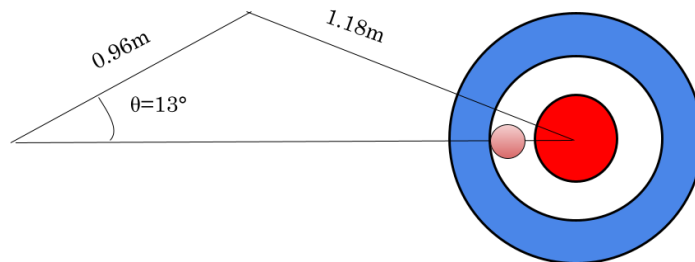


Consider the diagram above, which demonstrates that we must hit the second puck directly horizontal for the trajectory to be directly horizontal. We know that  $AC = 96$  cm and the angle is  $\angle CAE = 13^\circ$ . Project two points D, E directly below B, C respectively such that they are both collinear with A. Notice that triangles ABD and ACE share the same leg height, but have differing leg base lengths, which is the length of BC. Using trigonometric identities, we have that  $BD = CE = 96 \sin 13^\circ$  and  $AE = 96 \cos 13^\circ$ . Because D, E are collinear, and D lies in between A, E, then  $AD + DE = AE$ . But since DE is a direct projection of BC, then  $AD = AE - DE = AE - BC$ . So we have that  $AD = 96 \cos 13^\circ - 5.6 \approx 87.9$  and  $BD = 96 \sin 13^\circ \approx 21.6$ . Then, computing  $\angle BAD$ , which is our launch angle, we have that

$$\tan \angle BAD = \frac{BD}{AD} = \frac{21.6}{87.9} \implies \angle BAD \approx \boxed{13.8^\circ}$$

Answers within  $\pm 0.3^\circ$  are accepted.

- (d) [4 pts] You decide to try sliding the first puck itself straight for the target. If the table has  $\mu_k = \mu_s = \mu = 0.05$ , find the minimum force (in N), given contact time of  $t = 1.5$  s, from the bicep such that it lands in the white, scoring four points.

**Solution:**

The hard part of this solution isn't necessarily the concept of the problem, but finding the distance between the center of the puck to the center of the target, then subtracting to get the minimum distance. Refer to the diagram above for the rest of the solution. Let the distance needed to travel be  $x$ . We know that the frictional coefficient is  $\mu = 0.05$ , so we have a deceleration of  $a = -9.81 \cdot 0.05 = -0.49 \text{ m/s}^2$ . Using kinematics, we have that  $v_f^2 = v_i^2 + 2a\Delta d$ . Since the puck stops,  $v_f = 0$ , and we have  $\Delta d = x$ , so we have that  $v_f = \sqrt{0.98 \cdot x} \text{ m/s}$ . Now we seek to find  $x$ .

To do this, we find the distance between the puck and the center of the target, then subtract distance so it is as far away from the center while still being in the four-point region. Let the total distance be  $d$ . Then, using law of cosines, we have that

$$1.18^2 = 0.96^2 + d^2 - 2 \cdot 0.96 \cdot d \cdot \cos(13^\circ) \implies d = 2.095 \text{ m or } 2.10 \text{ m}.$$

Note that the negative solution can be discarded.

Now, consider the "at least" four-point region to be the white circle's overall radius (including the red circle center). Although we don't know this, we know the red circle has a radius of 6 cm, and the thickness of the white band is 8 cm, so the radius of the (at least) four-point circle must be 6 cm + 8 cm = 14 cm. However, for the puck to be completely inside the white circle, it must be at least  $r = 2.8$  cm away from the edge of the circle, or at most 14 cm - 2.8 cm = 11.2 cm from the center. Subtracting this off the total distance gives us

$$x = d - 0.112 \text{ m} = 2.10 \text{ m} - 0.112 \text{ m} \approx 1.98 \text{ m}.$$

Plugging this back into our original equation, we have  $v_f = \sqrt{0.98 \cdot x} \text{ m/s} \approx 1.39 \text{ m/s}$ . Using momentum equations with  $m = 0.045 \text{ kg}$  and  $\Delta t = 1.5 \text{ s}$ , we have  $F_{\text{handle}} = mv/\Delta t = 0.045 \cdot 1.39/1.5 = 0.0417 \text{ N}$ . To find the force from the bicep, we divide this by the IMA, giving us

$$F_{\text{bicep}} = \frac{F_{\text{handle}}}{\text{IMA}} = \frac{0.0417 \text{ N}}{0.124} = \boxed{0.336 \text{ N}}.$$

Answers within  $\pm 0.014 \text{ N}$  are accepted.

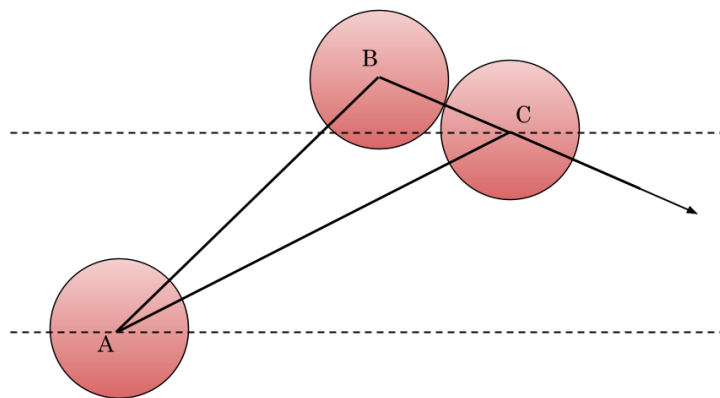
- (e) You decide to hit the second puck instead.
- [1 pt] How many degrees under the horizontal must the second puck travel to be dead center?
  - [3 pts] Find the initial angle of the first puck (relative to the horizontal) such that the second puck can land dead center.
  - [3 pts] At the angle you found in (ii), find the maximum velocity (in m/s) of the first puck such that the second puck is dead center.

**Solution:**

- i. We notice that the angle formed between the second puck, the center of the target, and the first puck is merely equal to the angle in question (construct parallel lines!). Using law of sines, we simply have that

$$\frac{0.96}{\sin(13^\circ)} = \frac{1.18}{\sin(\theta_2)} \implies \boxed{\theta_2 = 16.1^\circ}$$

Answers within  $\pm 0.3^\circ$  are accepted. Note that the other solution,  $\theta = 163.9^\circ$ , is discarded since we know that the angle is acute from the diagram.

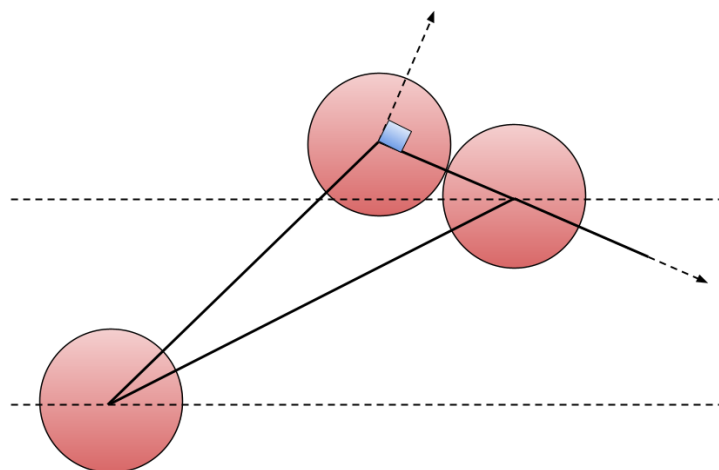


- ii. Refer to the first diagram above for this solution. Examining  $\angle BCA$ , we note that it is the sum of two angles; first, the angle below the 'second' horizontal, and the second which is the angle above the horizontal. The angle above the horizontal has the same angle as the trajectory angle of the second puck (they are vertical angles, convince yourself this is true!), which we have found to be  $\theta_2 = 16.1^\circ$ . The angle below the horizontal has the same angle as the angle between the original two pucks (the parallel horizontals and the line intersecting both), which is given to be  $13^\circ$ . This means that  $\angle BCA = 16.1^\circ + 13^\circ = 29.1^\circ$ .

Then, we note that  $BC = 5.6$ ,  $AC = 96$ . Using law of cosines, we have that  $AB^2 = BC^2 + AC^2 - 2 \cdot BC \cdot AC \cdot \cos(\angle BCA)$ , and plugging in our values, we have that  $AB^2 = (5.6)^2 + (96)^2 - 2 \cdot 5.6 \cdot 96 \cdot \cos(29.1^\circ) \implies AB = 0.911$  m. Then, using law of sines, we have that

$$\frac{\sin \angle BAC}{0.056} = \frac{\sin(29.1^\circ)}{0.911} \implies \angle BAC = 1.71^\circ$$

Then, because we are measuring this relative to the original  $13^\circ$  displacement between the two pucks, we add this to  $13^\circ$  to get a final angle of  $\boxed{\theta_1 = 14.7^\circ}$ . Answers within  $\pm 0.3^\circ$  are accepted.



- iii. Because collisions in this situation are *perfectly elastic* **and** *the masses of each puck are the same*, we know that the resultant angle between each puck's motion after collision is  $90^\circ$ . This means the trajectory of the first puck is  $90^\circ - 16.1^\circ = 73.9^\circ$  (above the horizontal). Let the original velocity of the first puck, the post-collision velocity of the first puck, and the resultant velocity of the second puck be  $v_1, v'_1, v_2$ . Because the second puck must travel 1.18 m, we have  $0 = v_2^2 + 2 \cdot -0.49 \cdot 1.18$ , giving us  $v_2 = 1.08 \text{ m/s}$ . Putting together horizontal and vertical momentum equations, we have that

$$v_1 \cos(14.7^\circ) = v'_1 \cos(73.9^\circ) + 1.08 \cos(16.1^\circ),$$

$$v_1 \sin(14.7^\circ) = v'_1 \sin(73.9^\circ) - 1.08 \sin(16.1^\circ).$$

Note there is a subtract on the vertical component because the second puck moves below relative to the horizontal. Solving this system nets us  $v_1 = 1.26 \text{ m/s}$  as our final answer. Answers within  $\pm 0.05 \text{ m/s}$  are accepted.

Remark: you can use the conservation of energy equation, but this also results from squaring each momentum equation and adding them.

Many thanks to the CMU pool players I met with (Sagar, Alex, Darren, and Colby) that helped me understand pool physics which I then converted to pucks for simplicity.

4. **Kid's Menu.** You know how restaurants have kid's menus with puzzles to keep them distracted?

Part	$A$	$B$	$C$	$D$	$E$	$F$
(a)	1	0	1	?	$\infty$	$\infty$
(b)	1	?	0	0	0	0
(c)	$\infty$	?	?	1	?	?
(d)	?	1	1	$\infty$	?	1

Use the diagram on the next page and the table above to answer the following questions. Assume all pulleys and strings are massless and frictionless and grey bars to be immovable. Any pulley directly connected to an immovable bar is translationally fixed. From left to right, the coaxial pulleys have diameter ratios of  $2 : 4 : 5$ ,  $1 : 3 : 5$ , and  $1 : 2$ . Assume the small angle effect on the topmost coaxial pulley is negligible. The value in the corresponding cell in the table is the mass of the box in units of beavers:  $1 \text{ bvr} = 1/9.81 \text{ kg}$ . Note that a mass of infinity ( $\infty$ ) indicates the box is immovable and a question mark indicates the mass of the box is to be determined. All masses are initially at rest and are released simultaneously.

- [2 pts] What is the mass of box  $D$ , in beavers, if the system is in equilibrium?
- [2 pts] What is the mass of box  $B$ , in beavers, if the system is in equilibrium?
- [3 pts] Identify all possible sets of masses ( $M_B, M_C, M_E, M_F$ ) such that the system is in equilibrium.
- Warning: These questions may be tedious...
  - [5 pts] Suppose  $M_A = M_E = M > 0$ . There is a range of masses  $M_{\min} < M < M_{\max}$  (in beavers) such that box  $C$  accelerates upwards.  $M_{\min/\max}$  can be expressed in the form  $(a \pm \sqrt{b})/c$  where  $a, b, c$  are relatively prime positive integers. Find them.
  - [5 pts] Let  $M = 1$ . Compute the rates and directions of acceleration of box  $A$  and  $E$ , in  $\text{m/s}^2$ .
  - [3 pts] After one second, what is the rate at which the potential energy of the whole system is being converted into kinetic, in W?
- [0 pts] Name the fellow in the corner.

### Solution:

- Balancing the leftmost coaxial pulley, the tension from the left, downwards string is  $1/3 \text{ N}$ . Then, the tension of the strings below and above  $C$  are  $1/6 \text{ N}$  and  $7/6 \text{ N}$ . Finally, the tension of the string above  $D$  is  $7/3 \text{ N}$  and therefore  $M_D = \boxed{7/3 \text{ bvr.}}$
- The IMA of the pulley system above  $B$  is 5. Balancing the leftmost coaxial pulley,  $M_B = \boxed{2/3 \text{ bvr.}}$
- Obviously, the mass of all boxes are non-negative. We proceed left to right. Since box  $A$  is immovable, box  $B$  can have any mass. Box  $D$  is in equilibrium, so the string connected to the top of box  $C$  has a tension of  $1/2 \text{ N}$  which implies  $M_C \leq 1/2 \text{ bvr}$ . Finally, the string connected to the top of the rightmost coaxial pulley has a tension of  $1/4 \text{ N}$ , so  $M_E + M_F \leq 1/4 \text{ bvr}$  and the torque balance on the coaxial pulley requires  $2M_E = M_F$ .

- (d) i. A Newtonian approach (i.e., using  $F = ma$ ) can be applied to solve this problem, but it will be tedious. We detail a more direct solution leveraging Lagrangian mechanics.

First, we note that the system has two degrees of freedom: the coordinate of box  $C$  and the coordinate of  $F$  relative to the rightmost coaxial pulley. Let these be  $x$  and  $y$  respectively. Then, the constraints imposed by the strings allow us to determine the positions (and thereby rates) of all other masses.

With this, we can write the system kinetic energy  $T$  and potential energy  $V$  as:

$$\begin{aligned} T &= \frac{1}{2g} \left[ M \left( \frac{3}{4} \dot{x} \right)^2 + \left( -\frac{1}{4} \dot{x} \right)^2 + \dot{x}^2 + M (-2\dot{y} - 2\dot{x})^2 + (\dot{y} - 2\dot{x})^2 \right] \\ &= \frac{1}{g} \left[ \left( \frac{73M + 81}{32} \right) \dot{x}^2 + (4M - 2)\dot{x}\dot{y} + \left( \frac{4M + 1}{2} \right) \dot{y}^2 \right], \\ V &= \frac{3}{4}Mx - \frac{1}{4}x + x + M(-2x - 2y) + (y - 2x) \\ &= -\frac{5}{4}(M + 1)x + (-2M + 1)y. \end{aligned}$$

Applying Euler–Lagrange, we produce the equations of motion:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{x}} + \frac{\partial V}{\partial x} &= \left( \frac{73M + 81}{16} \right) \ddot{x} + (4M - 2)\ddot{y} - \frac{5}{4}(M + 1) = 0, \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{y}} + \frac{\partial V}{\partial y} &= (4M + 1)\ddot{y} + (4M - 2)\ddot{x} + (-2M + 1) = 0. \end{aligned}$$

Solving the system of two equations, we find

$$\begin{aligned} \ddot{x} &= -12(4M^2 - 19M + 1)g/\Delta, \\ \ddot{y} &= (66M^2 + 49M - 41)g/\Delta, \end{aligned} \quad \text{where} \quad \Delta := 36M^2 + 653M + 17. \quad (1)$$

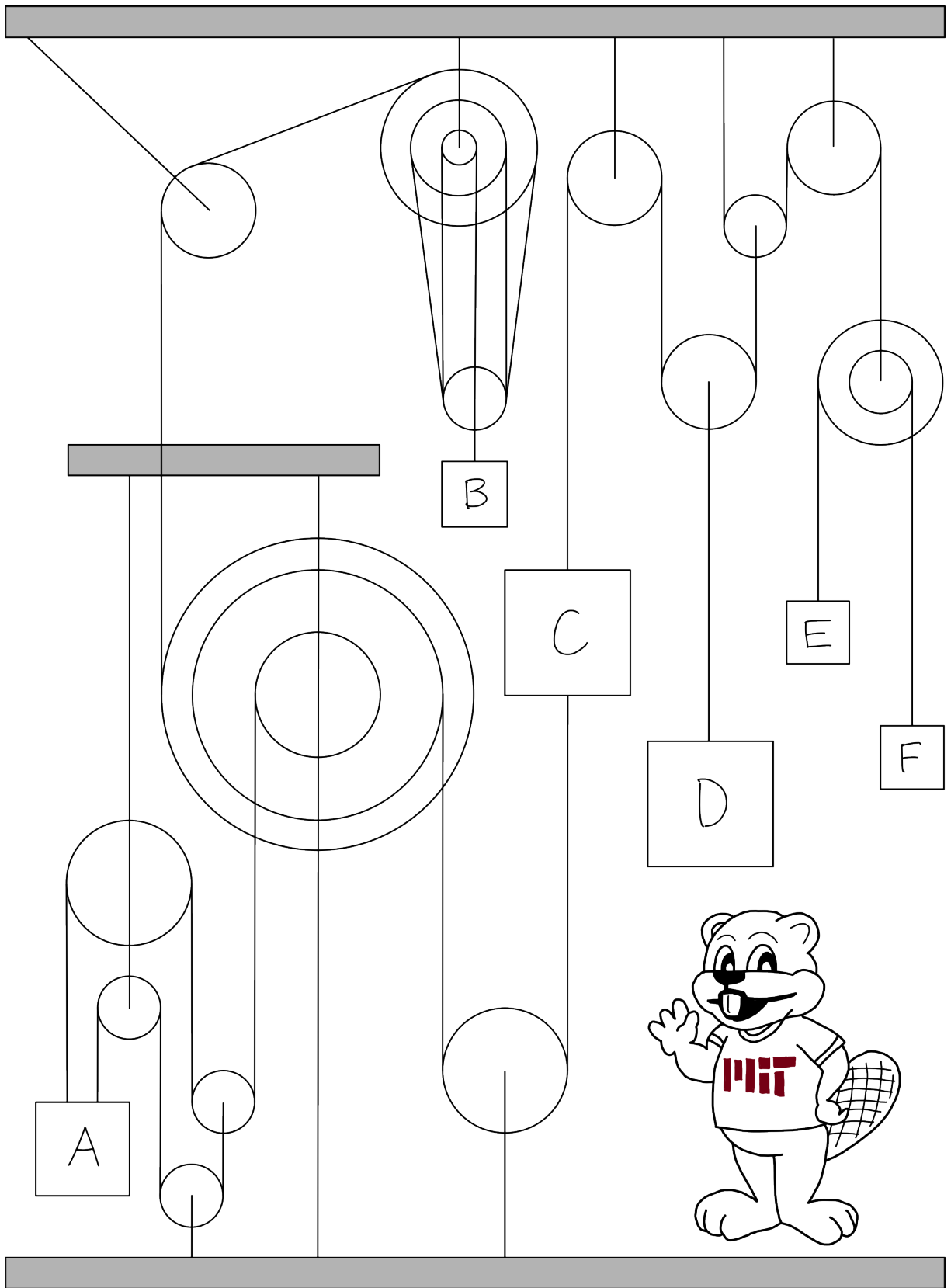
Since  $M > 0 \implies \Delta > 0$ , the condition  $\ddot{x} > 0$  is bounded by the roots of the numerator of  $\ddot{x}$ :

$$4M^2 - 19M + 1 = 0 \implies M_{\max/\min} = \boxed{\frac{19 \pm \sqrt{345}}{8} \text{ bvr.}}$$

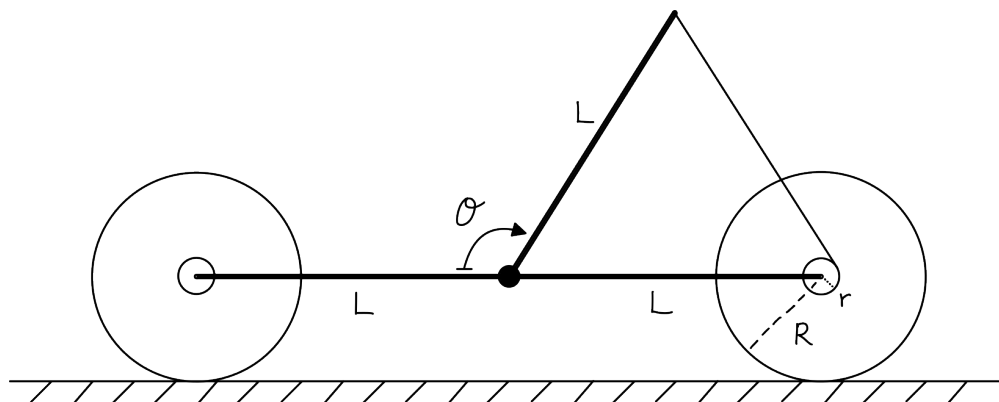
- ii. Plugging in  $M = 1$  to (1), we find  $\ddot{x} = 2.3344 \text{ m/s}^2$  and  $\ddot{y} = 1.0282 \text{ m/s}^2$ . The acceleration of boxes  $A$  and  $E$  are  $a_A = 3/4\ddot{x} = \boxed{1.7508 \text{ m/s}^2}$  (upwards) and  $a_E = -2\ddot{y} - 2\ddot{x} = \boxed{-6.7253 \text{ m/s}^2}$  (downwards).
- iii. After one second of constant acceleration, the velocities are  $\dot{x} = 2.3344 \text{ m/s}$  and  $\dot{y} = 1.0282 \text{ m/s}$ . The energy conversion rate is simply

$$|\dot{V}| = \left| \frac{5}{4}(M + 1)\dot{x} + (2M - 1)\dot{y} \right| = \boxed{6.8642 \text{ W.}}$$

- (e) Tim the beaver



5. **Mousetrap Vehicle.** Consider the diagram of a mousetrap vehicle below.



The vehicle is constructed from a rigid uniform plate (the chassis) of length  $2L$  and mass  $M$ , four massless wheels of equal radius  $R$  connected in pairs by two rigid massless axles of equal radius  $r$ , and a rigid massless lever of length  $L$  connected to the midpoint of the chassis by a torsion spring (the mousetrap) with a spring constant  $\kappa$  on one end and tied to a massless string on the other end that is spooled around the axle on the right such that it does not slip. The torsion spring on the lever imparts a linear-elastic torque in the counterclockwise direction; more formally:  $\tau = -\kappa\theta$ , where  $\theta$  is the angle the mousetrap is opened to. The static and kinetic coefficients of friction between the floor and the wheels are  $\mu_s = \mu_k = \mu$ . All other interactions are frictionless.

(a) [1 pt] What direction does the vehicle move? Right or left?

Express your answers to parts (b) through (d) in terms of the given variables and  $g$ .

(b) [2 pts] When the vehicle is at rest, what is the normal force on each wheel?

(c) [2 pts] What is the maximum horizontal acceleration of the vehicle without the wheels slipping?

(d) [4 pts] Derive the expression for the torque on the right axle as function of  $\theta$ .

Compute your answers to parts (e) and (f) in the required units using the parameters:  $L = 20$  cm,  $M = 700$  g,  $R = 7$  cm,  $r = 5$  mm,  $\kappa = 6$  N cm/rad, and  $\mu = 0.7$ . Note that the diagram is not to scale.

(e) [3 pts] Evaluate the maximum angle the mousetrap can be opened without the wheels slipping, in degrees.

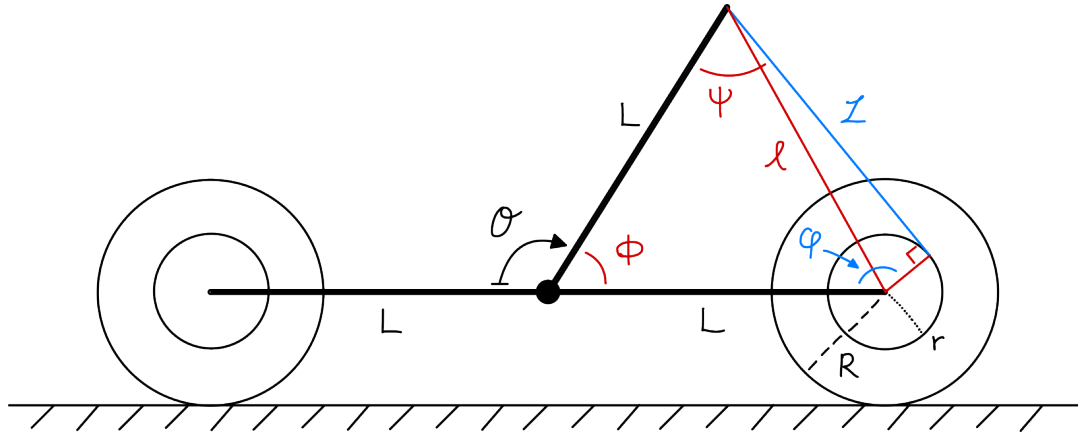
(f) [6 pts] Calculate the distance traveled if the mousetrap closes from  $100^\circ$  to  $15^\circ$ , in meters.

(g) [2 pts] Does the distance in part (f) change if each wheel has mass (e.g., 10 g)? Explain why or why not using qualitative arguments.



**Solution:**

- (a) Left
- (b) The weight of the vehicle is distributed evenly across the wheels. So  $Mg/4$ .
- (c) The driving force is applied through the rear (right) wheels; so the horizontal acceleration is limited to  $g\mu/2$ .



- (d) Let  $T$  be the tension in the string. Then the torque on the right axle is  $Tr$  since the string is tangent to the axle. We must determine the angle  $\psi$  as a function of  $\theta$  in order to solve for  $T$  with the torque balance equation  $TL \cos \psi - \kappa \theta = 0$ .

From the diagram above, we have  $\phi = \pi - \theta$  and  $\ell = 2L \sin(\phi/2)$ . With this, we find  $\psi = \theta/2 + \sin^{-1}(r/\ell)$ . Substituting, we get the expression

$$\tau_{\text{axle}}(\theta) = \kappa \theta \frac{r}{L} \sec \left[ \frac{\theta}{2} + \sin^{-1} \left( \frac{r}{2L} \csc \left[ \frac{\pi - \theta}{2} \right] \right) \right].$$

- (e) Balancing friction torque with the string torque gives  $RMg\mu/2 = \tau_{\text{axle}}$ . Solving this gives:  $148^\circ$ .
- (f) Since the angles are less than the critical angle from part (e), the vehicle moves without slipping and the distance traveled can be found exactly from the number of rotations of the right wheels. From part (d), we can recover the length of the unspooled string using the Pythagorean theorem:

$$\mathcal{L}(\theta) = \sqrt{4L^2 \sin^2 \left( \frac{\pi - \theta}{2} \right) - r^2}.$$

With this, we compute

$$\Delta \mathcal{L} := \mathcal{L}(15^\circ) - \mathcal{L}(100^\circ) = 13.9 \text{ cm}.$$

In addition to the rotation of the wheel contributed by the length of unspooled string, the tangent point of the string also rotates. That angle  $\varphi$  can be found using the angles we already have—

namely,  $2\pi = \phi + \psi + \varphi + \pi/2$ . So, we also compute

$$\Delta\varphi := \varphi(100^\circ) - \varphi(15^\circ) = 42.1^\circ \text{ or } 0.735 \text{ rad.}$$

So, the total distance traveled is

$$\Delta d = R \left( \frac{\Delta\mathcal{L}}{r} + \Delta\varphi \right) = \boxed{2.00 \text{ m.}}$$

We remark that if the added tangency rotation was omitted, the traveled distance would be 1.95 m.

- (g) No. Since the angles in part (f) are less than the critical angle from part (c), the vehicle moves without slipping. The number of rotations of the right wheels doesn't change and thereby the distance traveled does not as well.

One can also observe the added mass of the wheels would increase the maximum friction torque alongside decreasing the required friction torque for a given  $\tau_{\text{axle}}$  due to the required angular acceleration of the wheels. So the maximum mousetrap angle without slipping would increase. Notably, the added mass would increase the amount of time needed to travel  $\Delta d$ . This observation is not necessary for full credit.